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THE ELEMENTS

OF

PLANE TRIGONOMETRY,

WITH THE

CONSTRUCTION AND USE

OF

LOGARITHMIC TABLES OF NUMBERS,

AND THOSE OF

TRIGONOMETRIC FUNCTIONS OF ANGLES.

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THIRD EDITION.

CAMBRIDGE:
PRINTED BY JOHN W. PARKER, UNIVERSITY PRINTER.
T. STEVENSON, CAMBRIDGE;
A. H. BAILY & Co., LONDON.

M.DCCC.XXXVII.

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PLANE TRIGONOMETRY.

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1—17, (omit the Examples to 17.), 17—23, 26—28, 30, 32, 34, 35, 38—47, 49—56, 58, 69—76, 79, 82—84, 87—89, Exs. 1, 2, 3, 4, 90, 91, 99—106, (omit Cor.), 107, (omit Cor.), 108, 109, 112—122, 126, 134—136.

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PLANE TRIGONOMETRY.

CHAPTER I.

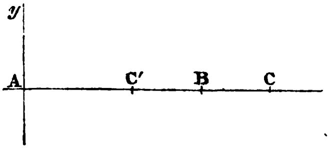
ON THE METHOD OF REPRESENTING LINES AND ANGLES, AND ON THE DIVISIONS OF ANGLES.

1. *The magnitudes of lines may be represented by algebraical quantities.*

To measure a line we find how many times it contains a fixed and definite line, as an inch or a yard, which has been previously fixed on as the unit of length. Thus, if an inch be taken as the unit of length, we say that a line is 30, if it contain an inch thirty times; and in like manner we call a line a, if it contain the unit of length a times.

2. *If lines drawn in a given direction from a fixed line be represented by positive quantities, then those which are drawn from that line in a contrary direction will be represented by negative quantities.*

If to a line AB measured from A towards the *right* of the fixed line Ay it be required to add a given line, it is evident that AB must be produced to a point C such that BC shall be equal to that given line,—and if from AB we have to take a given line, we must cut off from BA a part BC' equal to the given line, and AC' will be the required remainder.

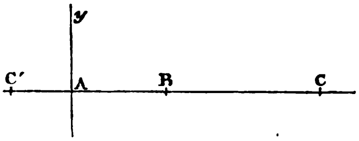


Let AB be represented by a, and BC by b, and make $BC' = BC$; then

$$AC' + C'B = AB;$$

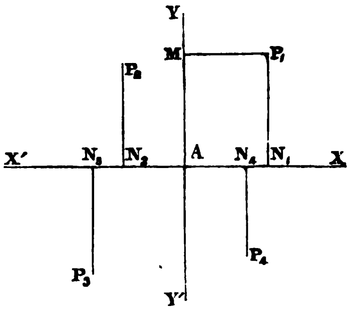
$$\therefore\ AC' = AB - C'B = a - b.$$

Now if b be greater than a, C' lies on the other side of A. And $AC' = a - b = -(b - a)$, which is a negative quantity equal in magnitude to the difference of the lines b and a; which difference we see lies in this case to the left of Ay.



Hence, if a line be represented by a negative quantity, it is meant that it is measured from Ay in a direction contrary to that in which the lines represented by positive quantities were measured.

3. Let XAX', YAY' be two fixed lines at right angles to each other, and produced, if necessary, indefinitely. Then the position of any point P_1 in the plane of these lines is known, if we know the magnitudes of the perpendiculars let fall from P_1 upon the lines XAX' and YAY', viz. P_1N_1 and P_1M (or AN_1, which is equal to P_1M).



AN_1, N_1P_1, are called *the co-ordinates of* P_1 *referred to the rectangular axes* AX *and* AY.

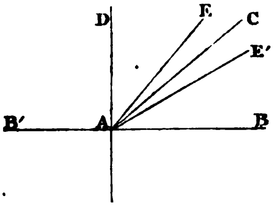
With respect to lines measured from A in the direction of the axis XAX', it is usual to call those *positive* which are measured to the *right* of the axis YAY', and therefore those will be *negative* which are measured to the *left* of YAY'.

With respect to lines measured in the direction of YAY' it is usual to call those *positive* which lie on the *upper* side of XAX', and therefore those will be *negative* which lie on the *lower* side of XAX'.

Thus if P_1 be a point in the space contained by AX and AY, its ordinate AN_1 which is measured in the direction of XAX' is *positive*, because it lies to the right of YAY'; and its ordinate N_1P_1 measured in the direction of YAY' is *positive*, because it lies on the upper side of XAX'. But if P_2 be the point, its ordinate AN_2 measured in the direction of XAX' is ***negative***, because it lies to the left of YAY'; and the ordinate N_2P_2 measured in the direction of YAY' is ***positive***, because it lies on the upper side of XAX'.

Similarly, the ordinates of P_3 and P_4 measured in the direction of XAX' are ***negative*** and ***positive*** respectively; and their ordinates measured in the direction of YAY' are ***negative*** in both cases.

4. Def. If a straight line revolve in one plane round its extremity A from a given position as AB into any other position as AC, the inclination of AC to AB is called *an angle* ($\angle$); and the angle is signified by the letters BAC or CAB, the middle letter being that placed at the point in which the two lines meet.



By continuing this revolving motion, we may suppose the angle to become of any magnitude whatever.

5. Def. If AD be equally inclined to the parts AB, AB' of the straight line BB', each of the angles BAD, $B'AD$ is called a ***right*** angle.

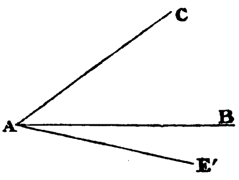
6. Def. An *acute* angle is less, and an ***obtuse*** angle is greater, than a ***right*** angle.

7. ***If the angles formed by* AC *revolving in one direction, as* BCD, *from* AB *be considered positive, then if* AC *revolve in the contrary direction from* AB *it will form negative angles.***

If to the angle BAC, Fig. Art. 4, it be required to add a given angle, CA must move in the direction BCD through an angle CAE equal to the given angle, and the whole BAE will be the angle required. And if it be required to take a given angle from BAC, CA must evidently move in the contrary direction till it come into a position $E'A$, such that $\angle CAE'$ is equal to the angle to be subtracted.

$$\text{Then } \angle CAE' + \angle E'AB = \angle BAC;$$

$$\therefore \angle E'AB = \angle BAC - \angle CAE'.$$



Now if $\angle CAE'$ be greater than $\angle CAB$, $E'A$ lies on the other side of AB,

$$\text{and } \angle E'AB = \angle BAC - \angle CAE'$$

$$= -(\angle CAE' - \angle BAC),$$

a negative quantity, whose magnitude is the difference between the angles CAE' and BAC; which difference, we see, lies in this case on the lower side of AB.

Hence, by calling an angle *negative*, we mean that it is formed by the revolving line moving in a direction contrary to that in which it revolved to trace out positive angles.

8. A right angle is divided by the English into 90 equal angles which are called degrees; a degree into 60 minutes; a minute into 60 seconds; a second into 60 thirds; and so on to fourths, fifths, &c. And in the same manner as the length of a given line is determined by finding how many times it contains the unit of length, (as a yard), and its subdivisions, (viz. feet and inches)—so the magnitude of an angle is determined by finding how many times it contains a degree and its subdivisions.

A degree and its subdivisions are thus marked, 24^0, $50'$, $34''$, $42'''$; which denotes an angle containing 24 degrees, 50 minutes, 34 seconds, and 42 thirds.

In practice the subdivisions of the angle beyond seconds are not commonly used; if very great accuracy be required, the subdivisions beyond seconds are expressed in decimal parts of a second: thus since

$$42''' = \frac{42}{60} \text{ of } 1'' = \frac{42''}{60} = \cdot 7'',$$

the angle above might have been written 24^0, $50'$, $34''\cdot 7$.

9. **By the French and other Continental Mathematicians,** a right angle is divided into 100 equal angles called **grades**; a grade into 100 minutes; a minute into 100 seconds; and so on; and the divisions are thus marked, 26^g, $24'$, $32''$, $47'''$.

$$\text{Now since}\quad 1' = \frac{1^g}{100} = \cdot 01^g$$

$$1'' = \frac{1'}{100} = \frac{1^g}{10000} = \cdot 0001^g$$

$$1''' = \frac{1''}{100} = \frac{1^g}{1000000} = \cdot 000001^g.$$

The above angle might have been written $26^g\cdot 243247$.

Whence it appears that if the French division be adopted, arithmetical operations can be performed on angles in the same manner as on any other decimal fractions; an advantage which does not attend the English division of the angle.

10. *To find the relation between* E *and* F, *the number of degrees and grades contained in the same angle* BAC. (Art. 14. Fig. 1.)

In the English division,

$$\frac{E^0}{90^0} = \frac{\text{given } \angle BAC}{\text{right } \angle};$$

In the French division,

$$\frac{F^g}{100^g} = \frac{\text{given } \angle BAC}{\text{right } \angle};$$

$$\therefore \frac{E}{90} = \frac{F}{100};$$

$$\therefore E = \frac{9}{10}F = F - \frac{F}{10},$$

$$\text{and } F = \frac{10}{9}E = E + \frac{E}{9}.$$

Ex. 1. To find how many degrees and minutes are contained in the angle $42^{g}, 34^{\backprime}, 56^{\backprime\backprime}$.

$$\begin{array}{rl}
F = & 42.3456 \\
\dfrac{F}{10} = & 4.23456 \\
\hline
\therefore\ E = F - \dfrac{F}{10} = & 38.11104 \\
& 60 \\
\hline
& 6.6624 \\
& 60 \\
\hline
& 39.744
\end{array}$$

$$\therefore\ E = 38^{0},\ 6',\ 39''.74,$$

retaining the tenths and hundredths and neglecting the *thousandth* parts of a second.

Ex. 2. Find how many grades and minutes are contained in the angle $24^{0}, 51', 45''$.

First reducing the minutes and seconds to the decimal parts of a degree,

$$\begin{array}{r}
60\big)\,45''. \\
\hline
60\big)\,51'.75 \\
\hline
.8625
\end{array}$$

$$\begin{array}{rl}
\therefore\ E = & 24.8625 \\
\dfrac{E}{9} = & 2.7625 \\
\hline
\therefore\ E + \dfrac{E}{9} = & 27.6250
\end{array}$$

$$\text{and } \therefore\ F = 27^{g},\ 62^{\backprime},\ 50^{\backprime\backprime}.$$

11. Def. *The complement of an angle is its defect from a right angle.*

Thus

$90^0 - 24^0, 32' = 65^0, 28'$ is the complement of $24^0, 32'$.

$90^0 - 110^0, 15' = -(20^0, 15')$ is the complement of $110^0, 15'$.

12. Def. *The supplement of an angle is its defect from two right angles.*

Thus $180^0 - 56^0, 20' = 123^0, 40'$ is the supplement of $56^0, 20'$.

$180^0 - 186^0, 12' = -(6^0, 12')$ is the supplement of $186^0, 12'$.

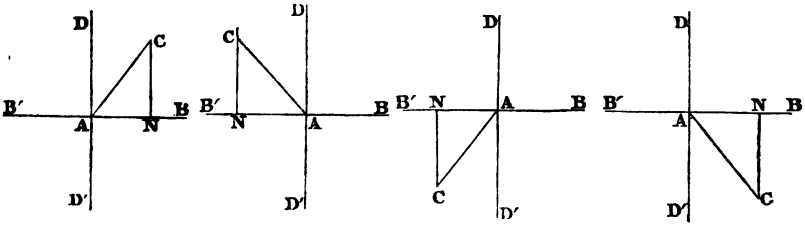
A few examples to each chapter are placed after the third appendix.

CHAPTER II.

OF GONIOMETRICAL FUNCTIONS OF ONE ANGLE, AND SOME FORMULÆ CONNECTING THEM WITH EACH OTHER.

13. Def. *Plane Trigonometry*, in its original meaning, implies *the measuring of plane triangles*;—in its extended signification, it treats of the formulæ connecting the relations of angles with each other, and of the solution of plane rectilineal figures.

14. Let a straight line revolve from the fixed line *AB* round the point *A*, in the direction of the letters *B*, *D*, *B′*, *D′*, and let there be *A* degrees in the angle *BAC*.



From any point *C* in *AC* draw *CN* at right angles to *AB*, produced if necessary, and through *A* draw *DAD′* perpendicular to *AB*.

Now, for the reasons given in Arts. 2 and 3, in these figures the signs of *NC* are +, +, −, − respectively, and those of *AN* are +, −, −, + respectively.

15. Definitions. See the figures of the last Article.

1. $\frac{NC}{AC}$ is the *sine* of the $\angle BAC$; or, $\sin \angle BAC = \frac{NC}{AC}$.

2. $\frac{AN}{AC}$ is the *cosine* of the $\angle BAC$; or, $\cos \angle BAC = \frac{AN}{AC}$.

3. $\frac{NC}{AN}$ is the *tangent* of the $\angle BAC$; or, $\tan \angle BAC = \frac{NC}{AN}$.

4. $\frac{AC}{AN}$ is the *secant* of the $\angle BAC$; or, $\sec \angle BAC = \frac{AC}{AN}$.

5. $1 - \cos \angle BAC$ is the *versed sine* of the $\angle BAC$;

$$\text{or, versin} \angle BAC = 1 - \cos \angle BAC;$$

6. The tangent of the complement of the $\angle BAC$ is called the *cotangent* of the $\angle BAC$;

$$\text{or, } \cot \angle BAC = \tan (90^0 - \angle BAC).$$

Cor. Since, if $90^0 - \angle BAC$ be the original angle, its complement is $\angle BAC$, Art. 11; we have by this definition,

$$\text{cotan} (90^0 - \angle BAC) = \tan \angle BAC,$$

$$\text{or, } \tan \angle BAC = \text{cotan} (90^0 - \angle BAC).$$

7. The secant of the complement of the $\angle BAC$ is called the *cosecant* of the $\angle BAC$;

$$\text{or, cosec} \angle BAC = \sec (90^0 - \angle BAC.)$$

Cor. Since if $90^0 - \angle BAC$ be the original angle, its complement is $\angle BAC$, Art. 11, we have by this definition,

$$\text{cosec} (90^0 - \angle BAC) = \sec \angle BAC;$$

$$\text{or, } \sec \angle BAC = \text{cosec} (90^0 - \angle BAC).$$

8. The versed sine of the supplement of the $\angle BAC$ is called the *suversine* of the $\angle BAC$;

$$\text{or, suversin} \angle BAC = \text{versin} (180^0 - \angle BAC).$$

16. The cosine of $\angle BAC$ might have been defined to be the sine of the complement of $\angle BAC$.

$$\text{For } \cos \angle BAC = \frac{AN}{AC} = \sin \angle ACN = \sin (90^0 - \angle BAC).$$

$$\text{Also, } \sin \angle BAC = \frac{NC}{AC} = \cos \angle ACN = \cos (90^0 - \angle BAC),$$

or the sine of an angle is equal to the cosine of its complement.

In the following pages we shall for the sake of convenience indicate an angle by a single letter, as $\sin A$, $\cos A$, $\tan B$.

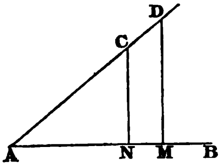
17. Sin A, cos A, tan A...*are proper measures to determine the magnitude of an angle, since so long as the angle* A *remains the same, they are invariable whatever be the magnitude of* AC.

Let D be any other point in AC, DM perpendicular to AB.

Then by definition,

$$\sin A = \frac{NC}{AC},$$

$$\text{or } \sin A = \frac{MD}{AD}.$$



But by similar triangles $\frac{NC}{AC} = \frac{MD}{AD}$; or $\sin A$ is the same wherever in the line AC the point C be situated;—similarly it may be shewn that $\cos A$, $\tan A$, $\sec A$...are invariable quantities so long as the magnitude of A remains unaltered.

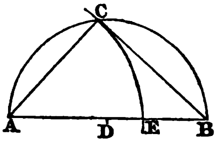
Hence, if any of the quantities $\sin A$, $\cos A$, $\tan A$, $\sec A$... be given, the angle A may be determined.

Ex. 1. Required to determine the angle whose sine is $\frac{2}{3}$.

Take any line AB and describe upon it a semicircle. From AB cut off the part BE equal to one third of AB, (Euclid VI. 9.), and with center A and radius AE describe a circle cutting ACB in C; join AC and CB.

Then the angle ACB, being in a semicircle, is a right angle, and

$$\sin ABC = \frac{CA}{AB} = \frac{AE}{AB} = \frac{2}{3};$$



Wherefore ABC is the angle required.

Ex. 2. Required the angle whose tangent is $\frac{4}{5}$.

Let a be any line and take $AN = 5a$, (fig. 1. Art. 20.)

Draw NC at right angles to AN, and let it $= 4a$: join AC.

$$\text{Then } \tan CAN = \frac{NC}{AN} = \frac{4a}{5a} = \frac{4}{5};$$

Wherefore NAC is the angle required.

18. Def. A quantity or expression of calculation is called *a function of a quantity*, if its value depend in any manner on the value of that quantity.

Since the magnitudes of the sine, the cosine, the tangent, &c. of an angle depend on the magnitude of the angle, they are *functions* of the angle; and when hereafter we make use of the term, "*the goniometrical functions* of an angle", we mean those functions which are defined in Art. 15.

19. *To express* versin A, cot A, cosec A, *in terms of the sides of the triangle* ANC. (Fig. 1. Art. 20.)

(1) $\text{Versin } A = 1 - \cos A = 1 - \frac{NA}{AC}$.

(2) $\text{Cot } A = \tan(90^0 - A)$

$= \tan ACN$

$= \frac{NA}{CN}$, by def. of the *tangent*.

(3) $\text{Cosec } A = \sec(90^0 - A)$

$= \sec ACN$

$= \frac{CA}{CN}$, by def. of the *secant*.

20. *To trace the variation in the* signs *of* sin A, cos A, tan A, sec A, *as* A *increases from* 0° to 360°.

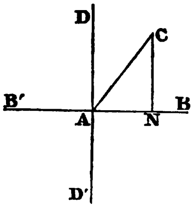
(1) $\text{Sin } A = \frac{NC}{AC}$, and therefore has the same sign in any case as NC has.

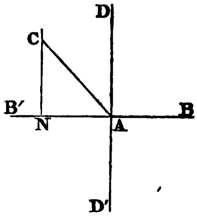
Hence, (Art. 14.) $\sin A$ is positive if A be between 0^0 and 180^0, (figs. 1, 2.); and it is negative if A be between 180^0 and 360^0. (Figs. 3, 4.)

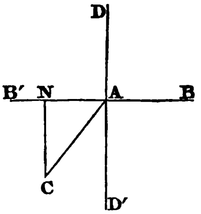
(2) $\text{Cos } A = \frac{AN}{AC}$, and therefore has the same sign as AN.

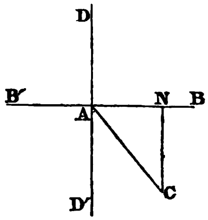
Hence, (Art. 14.) $\cos A$ is positive if A be between 0^0 and

90°, or between 270° and 360°, (figs. 1, 4.); and it is negative if A be between 90° and 270°. (Figs. 2, 3.)









(3) Tan $A = \dfrac{NC}{AN}$, and is therefore positive or negative according as NC and AN have the same or different signs.

Hence, (Art. 14.), tan A is positive, if A be between 0° and 90°, or 180° and 270°, (figs. 1, 3.); and it is negative, if A be between 90° and 180°, or 270° and 360°. (Figs. 2, 4.)

(4) Sec $A = \dfrac{AC}{AN}$, and therefore has the same sign as AN.

Hence sec A is positive if A be between 0° and 90°, or between 270° and 360°; and it is negative if A be between 90° and 270°.

21. *To trace the variations in the* magnitudes *of the sine, cosine, tangent, and secant, as the angle increases from* 0° *to* 360°. (Figs. Art. 20.)

Since (Art. 17.) the values of the sine, cosine, tangent, and secant are not affected by the magnitude of AC, we will suppose this line to remain of the same magnitude while $\angle A$ increases from 0° to 360°.

Now (fig. 1.) as AC revolves from the position AB into the position AD, NC increases from 0 to AC, and is positive; and AN decreases from AC to 0, and is positive.

As (fig. 2.) AC revolves from the position AD into the position AB', NC decreases from AC to 0, and is positive; and AN increases from 0 to AC, and is negative.

As (fig. 3.) AC revolves from AB' to AD', NC increases from 0 to AC, and is negative; and AN decreases from AC to 0, and is negative.

As (fig. 4.) AC revolves from AD' to AB, NC decreases from AC to 0, and is negative; and AN increases from 0 to AC, and is positive.

Hence we collect, that as

A changes from	0° to 90°	90° to 180°	180° to 270°	270° to 360°
Sin A $\left(\frac{NC}{AC}\right)$...	$\frac{0}{AC}$ to $\frac{+AC}{AC}$	$\frac{+AC}{AC}$ to $\frac{0}{AC}$	$\frac{0}{AC}$ to $\frac{-AC}{AC}$	$\frac{-AC}{AC}$ to $\frac{0}{AC}$
Cos A $\left(\frac{AN}{AC}\right)$...	$\frac{+AC}{AC} \dots \frac{0}{AC}$	$\frac{0}{AC} \dots \frac{-AC}{AC}$	$\frac{-AC}{AC} \dots \frac{0}{AC}$	$\frac{0}{AC} \dots \frac{+AC}{AC}$
Tan A $\left(\frac{NC}{AN}\right)$...	$\frac{0}{+AC} \dots \frac{+AC}{0}$	$\frac{+AC}{0} \dots \frac{0}{-AC}$	$\frac{0}{-AC} \dots \frac{-AC}{0}$	$\frac{-AC}{0} \dots \frac{0}{+AC}$
Sec A $\left(\frac{AC}{AN}\right)$...	$\frac{AC}{+AC} \dots \frac{AC}{0}$	$\frac{AC}{0} \dots \frac{AC}{-AC}$	$\frac{AC}{-AC} \dots \frac{AC}{0}$	$\frac{AC}{0} \dots \frac{AC}{+AC}$

These changes in sign and magnitude of the sine, cosine, tangent, and secant, may therefore be thus exhibited, the signs of the functions in each right angle being written in a bracket. The symbol ∞ represents an infinitely large quantity.

A being between }.	0° and 90°	90° and 180°	180° and 270°	270° and 360°
sin A......	0 and 1, (+)	1 and 0, (+)	0 and −1, (−)	−1 and 0, (−)
cos A......	1 ... 0, (+)	0 .. −1, (−)	−1 ... 0, (−)	0 ... 1, (+)
tan A......	0 ... ∞, (+)	∞ ... 0, (−)	0 ... ∞, (+)	∞ ... 0, (−)
sec A......	1 ... ∞, (+)	∞ .. −1, (−)	−1 ... ∞, (−)	∞ ... 1, (+)

Since cos A is never greater than unity, versin A (or $1 - \cos A$) is always positive; and its greatest value is when A becomes 180°, or cos A becomes -1, in which case versin A becomes 2.

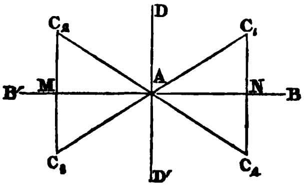
22. *To shew that*

$\sin A = \sin(180^\circ - A)$, *or* $= -\sin(180^\circ + A)$, *or* $= -\sin(360^\circ - A)$, *or* $= -\sin(-A)$.

Let $BAC_1 = A = B'AC_2 = B'AC_3 = BAC_4$;

and $AC_1 = AC_2 = AC_3 = AC_4$.

Join C_1C_4, and C_2C_3.



Then the angles at M and N are right angles, and NC_1, MC_2, MC_3, NC_4 are equal in magnitude; as are also AN and AM.

$$\text{Now, } \sin A = \frac{NC_1}{AC_1} = \frac{MC_2}{AC_2}$$
$$= \sin BAC_2$$
$$= \sin(BAD + DAB' - C_2AB')$$
$$= \sin(180^\circ - A) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\text{Again, } \sin A = \frac{NC_1}{AC_1} = \frac{-MC_3}{AC_3}, \text{ since } NC_1 = -MC_3,$$
$$= -\frac{MC_3}{AC_3}$$
$$= -\sin(BAD + DAB' + B'AC_3)$$
$$= -\sin(180^\circ + A) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\text{Again, } \sin A = \frac{NC_1}{AC_1} = \frac{-NC_4}{AC_4}$$
$$= -\frac{NC_4}{AC_4};$$

but $\frac{NC_4}{AC_4}$ is the sine either of the *positive* angle

$$(BAD + DAB' + B'AD' + D'AB - BAC_4),$$

or the sine of the *negative* angle BAC_4, (Art. 7.);

$$\therefore \sin A = -\sin(360^\circ - A) \ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$\text{or } = -\sin(-A) \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

23. In like manner it might be shewn that

$$\begin{aligned}\cos A &= -\cos(180^\circ - A),\\ \text{or } &= -\cos(180^\circ + A),\\ \text{or } &= \cos(360^\circ - A),\\ \text{or } &= \cos(-A).\end{aligned}$$

So also,

$$\begin{aligned}\tan A &= -\tan(180^\circ - A),\\ \text{or } &= \tan(180^\circ + A),\\ \text{or } &= -\tan(360^\circ - A),\\ \text{or } &= -\tan(-A).\end{aligned}$$

And,

$$\begin{aligned}\sec A &= -\sec(180^\circ - A),\\ \text{or } &= -\sec(180^\circ + A),\\ \text{or } &= \sec(360^\circ - A),\\ \text{or } &= \sec(-A).\end{aligned}$$

24. If any angle, as BAC_1, be increased by 360°, its sine will clearly remain unaltered, and therefore $\sin A$ is in all cases the same with $\sin(360^\circ + A)$. In like manner, $\sin(360^\circ + A) = \sin(2 \times 360^\circ + A)$, and so on. Therefore, if n be *any* positive integer,

$$\sin A = \sin.(n.360^\circ + A) = \sin(4n.90^\circ + A) \text{............} (1).$$

Similarly, $\sin A = \sin(180^\circ - A)$ Art. 22. (1).

$$\begin{aligned}&= \sin\{4n.90^\circ + (180^\circ - A)\}\\ &= \sin\{(4n+2)\,90^\circ - A\} \text{..................} (2).\end{aligned}$$

In like manner we get from (2) and (4) of Art. 22,

$$\sin A = -\sin\{(4n+2)\,90^\circ + A\} \text{..................} (3),$$

$$\sin A = -\sin(4n.90^\circ - A) \text{........................} (4).$$

25. By the same process of reasoning it may be proved from the formulæ of Art. 23, that

$$\begin{aligned}\cos A &= \cos(4n.90^\circ + A),\\ \text{or } &= -\cos\{(4n+2)\,90^\circ - A\},\\ \text{or } &= -\cos\{(4n+2)\,90^\circ + A\},\\ \text{or } &= \cos(4n.90^\circ - A).\end{aligned}$$

And,
$$
\begin{aligned}
\tan A &= \tan(4n\,.\,90^0 + A),\\
\text{or } &= -\tan\{(4n+2)\,.\,90^0 - A\},\\
\text{or } &= \tan\{(4n+2)\,.\,90^0 + A\},\\
\text{or } &= -\tan(4n\,.\,90^0 - A).
\end{aligned}
$$

Similarly it might be proved that,
$$
\begin{aligned}
\sec A &= \sec(4n\,.\,90^0 + A),\\
\text{or } &= -\sec\{(4n+2)\,.\,90^0 - A\},\\
\text{or } &= -\sec\{(4n+2)\,.\,90^0 + A\},\\
\text{or } &= \sec(4n\,.\,90^0 - A).
\end{aligned}
$$

26. From Arts. 16, 22, 23, we collect that

$$
\begin{cases}\sin A = \cos(90^0 - A)\\ \sin A = \sin(180^0 - A).\end{cases}
$$

$$
\begin{cases}\cos A = \sin(90^0 - A)\\ \cos A = -\cos(180^0 - A).\end{cases}
$$

$$
\begin{cases}\tan A = \cot(90^0 - A)\\ \tan A = -\tan(180^0 - A).\end{cases}
$$

$$
\begin{cases}\sec A = \operatorname{cosec}(90^0 - A)\\ \sec A = -\sec(180^0 - A).\end{cases}
$$

That is,

The sine of an angle = cosine of its complement,
or, = sine of its supplement.

The cosine of an angle = sine of its complement,
or, = – cosine of its supplement.

The tangent of an angle = cotangent of its complement,
or, = – tangent of its supplement.

The secant of an angle = cosecant of its complement,
or, = – secant of its supplement.

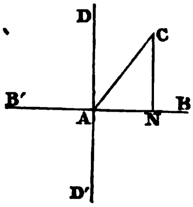
These relations between the sine, cosine, tangent, and secant, of an angle, and of its complement and supplement, will be continually made use of in the following pages.

27. The student will find it necessary to be perfectly familiar with the following expressions.

(1) $$\text{Tan}\,A = \frac{NC}{AN} = \frac{\dfrac{NC}{AC}}{\dfrac{AN}{AC}} = \frac{\sin A}{\cos A}.$$

(2) $$\text{Sec}\,A = \frac{AC}{AN} = \frac{1}{\dfrac{AN}{AC}} = \frac{1}{\cos A};$$

$$\therefore\ \cos A = \frac{1}{\sec A}.$$



(3) $$\text{Cot}\,A = \frac{AN}{NC} = \frac{\dfrac{AN}{AC}}{\dfrac{NC}{AC}} = \frac{\cos A}{\sin A}.$$

(4) $$\text{Cot}\,A = \frac{AN}{NC} = \frac{1}{\dfrac{NC}{AN}} = \frac{1}{\tan A};$$

$$\therefore\ \tan A = \frac{1}{\cot A}.$$

(5) $$\text{Cosec}\,A = \frac{AC}{NC} = \frac{1}{\dfrac{NC}{AC}} = \frac{1}{\sin A};$$

$$\therefore\ \sin A = \frac{1}{\text{cosec}\,A}.$$

(6) $$AC^2 = NC^2 + AN^2;$$

$$\therefore\ 1 = \left(\frac{NC}{AC}\right)^2 + \left(\frac{AN}{AC}\right)^2,$$

$$\text{or } 1 = (\sin A)^2 + (\cos A)^2;$$

$$\therefore \begin{cases} \sin A = \sqrt{1 - (\cos A)^2}, \\ \cos A = \sqrt{1 - (\sin A)^2}. \end{cases}$$

(7) $$AC^2 = AN^2 + NC^2;$$

$$\therefore \left(\frac{AC}{AN}\right)^2 = 1 + \left(\frac{NC}{AN}\right)^2,$$

$$\text{or } (\sec A)^2 = 1 + (\tan A)^2;$$

$$\therefore \begin{cases} \sec A = \sqrt{1 + (\tan A)^2}, \\ \tan A = \sqrt{(\sec A)^2 - 1}. \end{cases}$$

(8) $$AC^2 = AN^2 + NC^2;$$

$$\therefore \left(\frac{AC}{NC}\right)^2 = \left(\frac{AN}{NC}\right)^2 + 1,$$

$$\text{or } (\operatorname{cosec} A)^2 = (\operatorname{cotan} A)^2 + 1;$$

$$\therefore \begin{cases} \operatorname{cosec} A = \sqrt{1 + (\operatorname{cotan} A)^2}, \\ \operatorname{cotan} A = \sqrt{(\operatorname{cosec} A)^2 - 1}. \end{cases}$$

28. By means of the expressions proved in the last Article we can find the value of any one of the quantities defined in Art. 15, in terms of any other of those quantities.

(1) $$\operatorname{Tan} A = \frac{\sin A}{\sqrt{1 - (\sin A)^2}}.$$

$$\text{For } \tan A = \frac{\sin A}{\cos A}, \text{ Art. 27, (1),}$$

$$= \frac{\sin A}{\sqrt{1 - (\sin A)^2}}, \text{ (6).}$$

(2) $$\operatorname{Tan} A = \frac{\sin A}{\cos A} = \frac{\sqrt{1 - (\cos A)^2}}{\cos A}.$$

(3) $$\text{Sin}\,A = \frac{\sin A}{\cos A}.\cos A = \tan A.\frac{1}{\sec A}$$

$$= \frac{\tan A}{\sec A} = \frac{\tan A}{\sqrt{1+(\tan A)^2}}.$$

29. The formulæ proved in the last Article will often be found useful to the analyst. The same method of proof is applicable to all other questions of the same kind. Thus, required to express the cosine of an angle in terms of the cosecant, and the cosecant in terms of the versed-sine.

(1) $$\text{Cos}\,A = \sqrt{1-(\sin A)^2}$$

$$= \sqrt{1-\left(\frac{1}{\text{cosec}\,A}\right)^2}$$

$$= \frac{\sqrt{(\text{cosec}\,A)^2-1}}{\text{cosec}\,A}.$$

(2) $$\text{Cosec}\,A = \frac{1}{\sin A} = \frac{1}{\sqrt{1-(\cos A)^2}}$$

$$= \frac{1}{\sqrt{1-(1-\text{versin}\,A)^2}}$$

$$= \frac{1}{\sqrt{2\,\text{versin}\,A-(\text{versin}\,A)^2}}.$$

30. In the first of the following Tables those values of sin *A*, cos *A*, tan *A*, and sec *A*, are inserted, which are the most useful to be remembered; in the second Table are placed some others of less frequent occurrence in mathematical investigations:

1.

$\sin A$	$\cos A$	$\tan A$	$\sec A$
$\sqrt{\{1-(\cos A)^2\}}$ $\dfrac{\tan A}{\sqrt{\{1+(\tan A)^2\}}}$ $\dfrac{\sqrt{\{(\sec A)^2-1\}}}{\sec A}$	$\sqrt{\{1-(\sin A)^2\}}$ $\dfrac{1}{\sqrt{\{1+(\tan A)^2\}}}$ $\dfrac{1}{\sec A}$	$\dfrac{\sin A}{\sqrt{\{1-(\sin A)^2\}}}$ $\dfrac{\sqrt{\{1-(\cos A)^2\}}}{\cos A}$ $\sqrt{\{(\sec A)^2-1\}}$	$\dfrac{1}{\sqrt{\{1-(\sin A)^2\}}}$ $\dfrac{1}{\cos A}$ $\sqrt{\{1+(\tan A)^2\}}$

2.

$\sin A$	$\cos A$	$\tan A$	$\sec A$
$\dfrac{1}{\sqrt{\{1+(\cot A)^2\}}}$ $\dfrac{1}{\operatorname{cosec} A}$ $\sqrt{\{2\operatorname{versin} A-(\operatorname{versin} A)^2\}}$	$\dfrac{\cot A}{\sqrt{\{1+(\cot A)^2\}}}$ $\dfrac{\sqrt{\{(\operatorname{cosec} A)^2-1\}}}{\operatorname{cosec} A}$	$\dfrac{1}{\sqrt{\{(\operatorname{cosec} A)^2-1\}}}$ $\dfrac{\sqrt{\{2\operatorname{vers} A-(\operatorname{vers} A)^2\}}}{1-\operatorname{vers} A}$	$\dfrac{\sqrt{\{1+(\cot A)^2\}}}{\cot A}$ $\dfrac{1}{1-\operatorname{vers} A}$

31. *If* A *be less than* 45°, *or half a right angle, cos* A *is greater than sin* A. (Fig. Art. 27.)

Let $\angle NAC$ be less than 45°.

Then, since $\angle NAC + \angle NCA = 90^0$, $\angle NCA$ is greater than 45°. And in every triangle the greater side is opposite to the greater angle, (Eucl. I. 19.);

$$\therefore AN > NC;$$

$$\therefore \frac{AN}{AC} > \frac{NC}{AC},$$

$$\text{or } \cos A > \sin A.$$

Similarly, for angles between 45° and 90°, it may be shewn that the cosine is less than the sine.

32. *To find the sines, cosines, and tangents, of* 45°, 30°, *and* 60°.

(1) (Fig. Art. 27.) Let $\angle NAC = 45^0$;

$$\therefore \angle NCA = 90^0 - \angle NAC = 45^0;$$

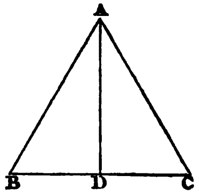
$$\therefore \frac{AN}{AC} = \frac{NC}{AC}; \quad \text{or } \sin 45^0 = \cos 45^0.$$

$$\text{Also } AC^2 = AN^2 + CN^2 = 2AN^2;$$

$$\therefore \sin 45^0 = \frac{AN}{AC} = \frac{1}{\sqrt{2}};$$

$$\therefore \cos 45^0 = \frac{1}{\sqrt{2}},$$

$$\text{and } \tan 45^0 = \frac{NC}{AC} = 1.$$



(2) Let *ABC* be an equilateral and equiangular triangle; therefore each of its angles is $\frac{1}{3}$ of two right angles, or 60°.

Let *AD* be perpendicular to *BC*;

$$\therefore\ BD = DC = \tfrac{1}{2}BC = \tfrac{1}{2}AB;\ \text{ and } \angle BAD = \angle DAC = 30^\circ;$$

$$\therefore\ \sin 30^\circ = \frac{DB}{AB} = \frac{\frac{1}{2}AB}{AB} = \frac{1}{2}.$$

$$\cos 30^\circ = \sqrt{\{1 - (\sin 30)^2\}} = \sqrt{(1 - \tfrac{1}{4})} = \frac{\sqrt{3}}{2},$$

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1}{\sqrt{3}}.$$

$$(3)\ \sin 60^\circ = \cos(90^\circ - 60^\circ),\ (\text{Art. } 16.) = \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

$$\cos 60^\circ = \sin(90^\circ - 60^\circ) = \sin 30^\circ = \frac{1}{2},$$

$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}.$$

33. The relations established in Art. 27, between different functions of the same angle, will frequently enable us to solve equations of the following kind.

Ex. 1. From the equation, $(\sin A)^2 + 5(\cos A)^2 = 3$, required the value of $\sin A$.

Since $(\cos A)^2 = 1 - (\sin A)^2$, the equation becomes

$$(\sin A)^2 + 5 \,.\, \{1 - (\sin A)^2\} = 3;$$

$$\therefore\ 4(\sin A)^2 = 2,\ \text{ and } \sin A = \frac{1}{\sqrt{2}}.$$

Ex. 2. $(\sin A)^2 = a \,.\, \cos A + b$, to determine $\cos A$.

If x be put for $\cos A$, $(\sin A)^2 = 1 - x^2$, and $1 - x^2 = ax + b$ will be the equation to be solved.

Ex. 3. The next example involves *two* unknown quantities.

From the equations

$$\sin A = m \,.\, \sin B,$$

$$\tan A = n \,.\, \tan B,$$

required the values of $\sin A$ and $\cos B$.

For $\sin A$ put x, and for $\cos B$ put y.

Then $\sin B = \sqrt{1 - y^2}$,

$$\tan A = \frac{\sin A}{\sqrt{\{1 - (\sin A)^2\}}} = \frac{x}{\sqrt{1 - x^2}}, \text{ (Art. 30.)},$$

$$\tan B = \frac{\sqrt{\{1 - (\cos B)^2\}}}{\cos B} = \frac{\sqrt{1 - y^2}}{y}, \text{ (Art. 30.)}$$

Making these substitutions, the equations become

$$\left\{\begin{array}{c} x = m\sqrt{1 - y^2} \\ \dfrac{x}{\sqrt{1 - x^2}} = n \,.\, \dfrac{\sqrt{1 - y^2}}{y} \end{array}\right. ;$$

from which, x and y may be determined.

$$\text{Ans.} \left\{\begin{array}{l} \sin A = \sqrt{\dfrac{m^2 - n^2}{1 - n^2}} \\ \cos B = \dfrac{n}{m}\sqrt{\dfrac{1 - m^2}{1 - n^2}} \end{array}\right\}.$$

Ex. 4. If $\left\{\begin{array}{l} m = \operatorname{cosec} A - \sin A \\ n = \sec A - \cos A \end{array}\right\}$, required the relation which exists between m and n; that is, required to find an equation between m and n, in which no function of the angle A shall enter.

From the first equation, $m = \dfrac{(\cos A)^2}{\sin A}$,

.... second ...., $n = \dfrac{(\sin A)^2}{\cos A}$;

$$\therefore\ \frac{n}{m} = \frac{(\sin A)^3}{(\cos A)^3} = (\tan A)^3;$$

$$\therefore\ \tan A = \frac{n^{\frac{1}{3}}}{m^{\frac{1}{3}}}.$$

$$\text{Also } m^2 = \frac{(\cos A)^4}{(\sin A)^2} = \frac{(\cos A)^2}{(\tan A)^2} = \frac{1}{(\tan A)^2 . (\sec A)^2},$$

$$\therefore\ m^2 . (\tan A)^2 . \{1 + (\tan A)^2\} = 1,$$

and by substitution,

$$m^{\frac{2}{3}} . n^{\frac{2}{3}} . (m^{\frac{2}{3}} + n^{\frac{2}{3}}) = 1.$$

CHAPTER III.

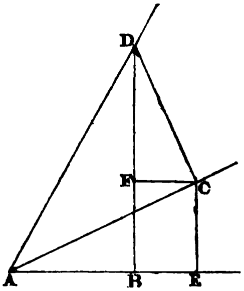
GONIOMETRICAL FORMULÆ INVOLVING MORE THAN ONE ANGLE.

34. *Given the sines and cosines of two angles, to find the sines and cosines of the angles equal to their sum and to their difference.*

BAC, *CAD* are two angles containing A^0 and B^0 respectively.

From *D*, any point in *AD*, draw *DB*, *DC* perpendiculars on *AB* and *AC*; and from *C* draw *CE*, *CF* perpendiculars on *AB* and *DB*.

Then *FE* is a rectangle; $FB = CE$, and $FC = BE$.



$\angle CDF = 90^0 - \angle DCF = \angle FCA = A$, since *CF* is parallel to *AE*.

$$\text{Now, } \sin(A+B) = \frac{BD}{AD} = \frac{BF+FD}{AD} = \frac{EC}{AD} + \frac{FD}{AD}$$

$$= \frac{EC}{AC}\cdot\frac{AC}{AD} + \frac{FD}{DC}\cdot\frac{DC}{AD}$$

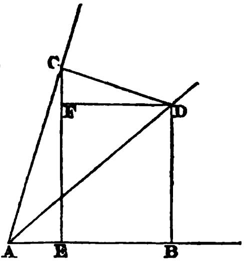
$$= \sin A \,.\, \cos B + \cos A \,.\, \sin B \ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{Also, } \cos(A+B) = \frac{AB}{AD} = \frac{AE-EB}{AD} = \frac{AE}{AD} - \frac{FC}{AD}$$

$$= \frac{AE}{AC}\cdot\frac{AC}{AD} - \frac{FC}{CD}\cdot\frac{CD}{AD}$$

$$= \cos A \,.\, \cos B - \sin A \,.\, \sin B \ldots\ldots\ldots\ldots\ldots(2).$$

Again, let $\angle BAC = A$, and $\angle CAD = B$.



From D, any point in AD, draw DB and DC perpendiculars on AB and AC, CE a perpendicular from C on AB, DF perpendicular to CE.

FB is a rectangle; $FE = DB$, and $FD = EB$;

$$\angle DCF = 90^0 - \angle ACE = A.$$

$$\text{Then, } \sin(A - B) = \frac{BD}{AD} = \frac{EC - CF}{AD} = \frac{EC}{AD} - \frac{CF}{AD}$$

$$= \frac{EC}{AC} \cdot \frac{AC}{AD} - \frac{CF}{CD} \cdot \frac{CD}{AD}$$

$$= \sin A . \cos B - \cos A . \sin B \ldots\ldots\ldots\ldots (3).$$

$$\text{Also, } \cos(A - B) = \frac{AB}{AD} = \frac{AE + EB}{AD} = \frac{AE}{AD} + \frac{FD}{AD}$$

$$= \frac{AE}{AC} \cdot \frac{AC}{AD} + \frac{FD}{CD} \cdot \frac{CD}{AD}$$

$$= \cos A . \cos B + \sin A . \sin B \ldots\ldots\ldots\ldots (4).$$

Ex. Having given the sines and cosines of 45° and 30°, required the sine and cosine of 75°, and of 15°.

$$\text{Sin } 45^0 = \cos 45^0 = \frac{1}{\sqrt{2}}, \ \sin 30^0 = \tfrac{1}{2}, \ \cos 30^0 = \frac{\sqrt{3}}{2}, \text{ Art. 32.}$$

$$\sin 75^0 = \sin(45^0 + 30^0) = \sin 45^0 . \cos 30^0 + \cos 45^0 . \sin 30^0$$

$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}$$

$$= \frac{1}{2\sqrt{2}} . (\sqrt{3} + 1);$$

$$\cos 75^0 = \cos(45^0 + 30^0) = \cos 45^0 . \cos 30^0 - \sin 45^0 . \sin 30^0$$

$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} . \frac{1}{2}$$

$$= \frac{1}{2\sqrt{2}} (\sqrt{3} - 1);$$

$$\sin 15^0 = \sin (45^0 - 30^0) = \sin 45^0 . \cos 30^0 - \cos 45^0 . \sin 30^0$$

$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} . \frac{1}{2}$$

$$= \frac{1}{2\sqrt{2}} (\sqrt{3} - 1) ;$$

$$\cos 15^0 = \cos (45^0 - 30^0) = \cos 45^0 . \cos 30^0 + \sin 45^0 . \sin 30^0$$

$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}$$

$$= \frac{1}{2\sqrt{2}} (\sqrt{3} + 1).$$

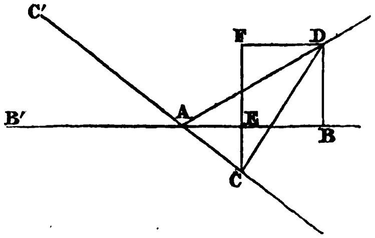
35. In the figures attached to the last Article, each of the simple angles A and B was represented as *less* than a right angle. But of whatever magnitude these simple angles are, if the same construction be made, and proper attention be paid to the signs of the sines and cosines of A and B, the same result will invariably be arrived at. For example, let it be required to prove the formula

$$\sin (A - B) = \sin A . \cos B - \cos A . \sin B$$

from the annexed figure, where $BAC' = A$, and $C'AD = B$; each angle being greater than a right angle.

From D, any point in AD, draw DC perpendicular to $C'A$ produced; let CEF be perpendicular to AB, DF parallel to AB, DB perpendicular to AB.

Then FB is a parallelogram, and $EF = DB$.



Now, $\sin (A - B) = \dfrac{BD}{AD} = \dfrac{EF}{AD}$

$$= \frac{CF - CE}{AD}$$

$$= \frac{CF}{CD} . \frac{CD}{AD} - \frac{CE}{AC} . \frac{AC}{AD}$$

$$= \cos FCD . \sin DAC - \sin CAE . \cos DAC,$$

but $\cos FCD = \cos EAC = \cos C'AB'$

$$= \cos (180^0 - C'AB) = -\cos A, \text{ Art. } 26.$$

$$\sin DAC = \sin (180^0 - DAC), \text{ Art. } 26, = \sin C'AD = \sin B,$$

$$\sin CAE = \sin C'AB' = \sin (180^0 - A) = \sin A,$$

$$\cos DAC = -\cos (180^0 - DAC) = -\cos B;$$

$$\therefore \sin (A-B) = -\cos A . \sin B + \sin A . \cos B,$$

$$= \sin A . \cos B - \cos A . \sin B.$$

36. *If one of the formulæ, as* $\sin (A + B) = \sin A . \cos B + \cos A . \sin B$, *be given, the others may be deduced from it.*

For let B become $(-B)$, then

$$\sin (A-B) = \sin \{A + (-B)\} = \sin A . \cos (-B) + \cos A . \sin (-B).$$

But $\cos (-B) = \cos B$, Art. 23;

and $\sin (-B) = -\sin B$, Art. 22;

$$\therefore \sin (A - B) = \sin A . \cos B - \cos A . \sin B.$$

Again,

$$\cos (A + B) = \sin \{90^0 - (A + B)\}, \text{ Art. } 16.$$
$$= \sin \{(90^0 - A) + (-B)\}$$
$$= \sin (90^0 - A) . \cos (-B) + \cos (90^0 - A) . \sin (-B).$$

But $\sin (90^0 - A) = \cos A$, and $\cos (90^0 - A) = \sin A$; Art. 16.

$$\therefore \cos (A + B) = \cos A . \cos B - \sin A . \sin B.$$

Again,

$$\cos (A - B) = \sin \{90^0 - (A - B)\}$$
$$= \sin \{(90^0 - A) + B\}$$
$$= \sin (90^0 - A) . \cos B + \cos (90^0 - A) . \sin B$$
$$= \cos A . \cos B + \sin A . \sin B.$$

37. From the formulæ of Art. 34, we can easily find the sine or cosine of the sum of three or more angles, in terms of the sines and cosines of the simple angles.

Given the sines and cosines of the angles A, B, C, *required the sine of* (A + B + C).

$$\sin(A + B + C) = \sin\{(A + B) + C\},$$

and, considering $(A + B)$ as one angle, this becomes

$$\sin(A + B) . \cos C + \cos(A + B) . \sin C$$

$$= (\sin A . \cos B + \cos A . \sin B) . \cos C + (\cos A . \cos B - \sin A . \sin B) . \sin C$$

$$= \sin A . \cos B . \cos C + \sin B . \cos A . \cos C + \sin C . \cos A . \cos B$$

$$- \sin A . \sin B . \sin C \ldots\ldots\ldots\ldots\ldots (1).$$

In like manner $\sin(A \pm B \pm C)$, and $\cos(A \pm B \pm C)$, may be found in terms of the sines and cosines of A, B, C; and the same method may be applied to the sum of any number of simple angles.

Cor. If $A + B + C = (2n + 1) . 180^0$, where n is an integer; since $\sin(2n + 1) . 180^0 = 0$, the equation (1) becomes

$$\sin A . \sin B . \sin C = \sin A . \cos B . \cos C$$

$$+ \sin B . \cos A . \cos C + \sin C . \cos A . \cos B \ldots\ldots\ldots (2).$$

If $n = 0$, $A + B + C = 180^0$, and therefore equation (2) expresses a relation which exists between the sines and cosines of the three angles of any plane triangle.

38. *To shew that* $\sin 2\text{A} = 2 \sin \text{A} \,.\, \cos \text{A}$.

$$\text{Sin}\,(A + B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B,$$

and writing A for B this becomes

$$\sin 2A = \sin A \,.\, \cos A + \cos A \,.\, \sin A = 2 \sin A \,.\, \cos A.$$

39. *To shew that* $\begin{cases} (1) \;\; \cos 2A = (\cos A)^2 - (\sin A)^2. \\ (2) \;\; \cos 2A = 2(\cos A)^2 - 1. \\ (3) \;\; \cos 2A = 1 - 2(\sin A)^2. \end{cases}$

(1) $\text{Cos}\,(A + B) = \cos A \,.\, \cos B - \sin A \,.\, \sin B,$

and writing A for B,

$$\cos 2A = \cos A \,.\, \cos A - \sin A \,.\, \sin A = (\cos A)^2 - (\sin A)^2.$$

Again,

$$\cos 2A = (\cos A)^2 - (\sin A)^2,$$

$$\text{And } 1 = (\cos A)^2 + (\sin A)^2;$$

$$\therefore \; 1 + \cos 2A = 2(\cos A)^2,$$

$$1 - \cos 2A = 2(\sin A)^2.$$

(2) Therefore, $\cos 2A = 2(\cos A)^2 - 1.$

(3) And, $\cos 2A = 1 - 2(\sin A)^2.$

40. *To shew that* $\begin{cases} \cos \text{A} + \sin \text{A} = \pm \sqrt{(1 + \sin 2\text{A})}, \\ \cos \text{A} - \sin \text{A} = \pm \sqrt{(1 - \sin 2\text{A})}. \end{cases}$

$$\text{Sin}\, 2A = 2 \sin A \,.\, \cos A,$$

$$\text{And } 1 = (\cos A)^2 + (\sin A)^2,$$

$\therefore$ by addition and subtraction,

$$1 + \sin 2A = (\cos A)^2 + 2 \sin A \,.\, \cos A + (\sin A)^2,$$

$$1 - \sin 2A = (\cos A)^2 - 2 \sin A \,.\, \cos A + (\sin A)^2;$$

$$\therefore \; \cos A + \sin A = \pm \sqrt{(1 + \sin 2A)},$$

$$\text{and } \cos A - \sin A = \pm \sqrt{(1 - \sin 2A)}.$$

41. *To shew that if* A *be less than* 45°, *then*

$$\begin{cases}\cos A = \frac{1}{2}\{\sqrt{(1+\sin 2A)}+\sqrt{(1-\sin 2A)}\},\\ \sin A = \frac{1}{2}\{\sqrt{(1+\sin 2A)}-\sqrt{(1-\sin 2A)}\}.\end{cases}$$

By Art. 31, if $A < 45°$, $\cos A$ is $> \sin A$, and they are both positive; therefore $\cos A + \sin A$ and $\cos A - \sin A$ are both positive quantities when A is $< 45°$;

$$\therefore\ \cos A + \sin A = +\sqrt{(1+\sin 2A)},$$
$$\text{and } \cos A - \sin A = +\sqrt{(1-\sin 2A)}.$$

∴ by addition and subtraction,

$$2\cos A = \sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)},$$
$$2\sin A = \sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)};$$
$$\therefore\ \begin{cases}\cos A = \frac{1}{2}\{\sqrt{(1+\sin 2A)}+\sqrt{(1-\sin 2A)}\},\\ \sin A = \frac{1}{2}\{\sqrt{(1+\sin 2A)}-\sqrt{(1-\sin 2A)}\}.\end{cases}$$

42. *The equations deduced in* Art. 40 *can be applied in any case to determine the sine and cosine of* A *from the sine of* 2 A.

For example, if A be an angle $> 3 \times 45°$ but $< 4 \times 45°$, (*i. e.* any angle comprehended under the form $180° - B$, where B is $< 45°$), $\cos A$ is negative, and greater in magnitude than $\sin A$, which is a positive quantity.

In this case therefore,

$$\cos A + \sin A = -\sqrt{(1+\sin 2A)},$$
$$\cos A - \sin A = -\sqrt{(1-\sin 2A)};$$
$$\therefore\ \cos A = -\tfrac{1}{2}\{\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}\},$$
$$\sin A = \tfrac{1}{2}\{\sqrt{(1-\sin 2A)} - \sqrt{(1+\sin 2A)}\}.$$

If at first sight this value of $\sin A$ appear to be negative, it is to be remembered that $\sin 2A$ is a negative quantity, ($2A$ being between 270° and 360°) and therefore $1 - \sin 2A$ is greater than $1 + \sin 2A$,—and therefore the value of $\sin A$ is here a positive quantity, as it ought to be.

43. *Given the tangents of two angles, to find the tangents of their sum and their difference.*

$$\text{Tan}\,(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \,.\, \cos B + \cos .\, A \sin B}{\cos A \,.\, \cos B - \sin .\, A \sin B},$$

and dividing the numerator and the denominator by $\cos A \,.\, \cos B$,

$$\tan(A+B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \,.\, \dfrac{\sin B}{\cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}.$$

Similarly, $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \,.\, \tan B}$.

Cor. 1. If $B = A$, $\tan 2A = \dfrac{2\tan A}{1 - (\tan A)^2}$.

Cor. 2. If $B = 45^0$, since $\tan 45^0 = 1$, (Art. 32.)

$$\tan(A + 45^0) = \frac{1 + \tan A}{1 - \tan A} \text{........................(1).}$$

$$= \frac{1 + \dfrac{\sin A}{\cos A}}{1 - \dfrac{\sin A}{\cos A}}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A} \text{..................(2).}$$

Cor. 3. Similarly, $\tan(A - 45^0) = \dfrac{\tan A - 1}{\tan A + 1}$ ....(3);

$$\text{or} = \frac{\sin A - \cos A}{\sin A + \cos A} \text{......(4);}$$

Cor. 4. $$\operatorname{Tan}(A+45^0)+\tan(A-45^0)=\frac{1+\tan A}{1-\tan A}+\frac{\tan A-1}{\tan A+1}$$

$$=\frac{4\tan A}{1-(\tan A)^2}$$

$$=2\tan 2A,\quad \text{by Cor. 1}\ldots\ldots\ldots\ldots\ldots(5).$$

If A be $< 45^0$;

Since $\tan(A-45^0)=\tan-(45^0-A)=-\tan(45^0-A)$,

(Art. 23.)

the last expression becomes,

$$\tan(45^0+A)-\tan(45^0-A)=2\tan 2A\ldots\ldots\ldots(6).$$

44. *To shew that* $\dfrac{\tan A+\tan B}{\tan A-\tan B}=\dfrac{\sin(A+B)}{\sin(A-B)}$.

$$\frac{\tan A+\tan B}{\tan A-\tan B}=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}$$

$$=\frac{\sin A.\cos B+\cos A.\sin B}{\sin A.\cos B-\cos A.\sin B}$$

$$=\frac{\sin(A+B)}{\sin(A-B)}.$$

45. *Given* tan A, tan B, tan C, *to find* tan (A + B + C).

$$\tan(A+B+C)=\tan\{(A+B)+C\}$$

$$=\frac{\tan(A+B)+\tan C}{1-\tan(A+B).\tan C},\quad \text{(Art. 43.)},$$

$$=\frac{\dfrac{\tan A+\tan B}{1-\tan A.\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A.\tan B}.\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A.\tan B.\tan C}{1-\tan A.\tan B-\tan A.\tan C-\tan B.\tan C}.$$

Cor. If $A + B + C = (2n + 1) . 180^0$, n being an integer,

$$\tan(A + B + C) = 0;$$

$$\therefore \tan A + \tan B + \tan C - \tan A . \tan B . \tan C = 0,$$

$$\text{or,} \quad \tan A + \tan B + \tan C = \tan A . \tan B . \tan C.$$

And since, if $n = 0$, $A + B + C = 180^0$, this relation between the tangents of A, B, C, is that which obtains between the tangents of the angles of any plane triangle.

In the same manner the tangent of the sum of four or more angles might be found in terms of the tangents of the simple angles.

46. *To find the values of* sin 2 A, *and* cos 2 A, *in terms of* tan A.

$$\operatorname{Sin} 2A = 2 \sin A . \cos A, \quad \text{Art. 38,}$$

$$= \frac{2 \sin A}{\cos A} . (\cos A)^2$$

$$= \frac{2 \tan A}{(\sec A)^2}, \quad \text{Art. 27, (2),}$$

$$= \frac{2 \tan A}{1 + (\tan A)^2}, \quad \text{Art. 27, (7).}$$

Again, $\cos 2A = 2 (\cos A)^2 - 1$, Art. 39, (2),

$$= \frac{2}{(\sec A)^2} - 1$$

$$= \frac{2}{1 + (\tan A)^2} - 1$$

$$= \frac{1 - (\tan A)^2}{1 + (\tan A)^2}.$$

47. The following values of $\sin 2A$, $\cos 2A$, $\tan 2A$, are of frequent occurrence, and necessary to be remembered; those of them which have not been proved already are easily deduced after the method used in the last Article.

1. $\operatorname{Sin} 2A = 2 \sin A . \cos A.$

2. $\ldots\ldots = \dfrac{2 \tan A}{1 + (\tan A)^2}.$

3. $\ldots\ldots = \dfrac{2 . \sqrt{\{(\sec A)^2 - 1\}}}{(\sec A)^2}.$

4. $\operatorname{Tan} 2A = \dfrac{2 \tan A}{1 - (\tan A)^2}.$

1. $\operatorname{Cos} 2A = (\cos A)^2 - (\sin A)^2.$

2. $\ldots\ldots = 2 (\cos A)^2 - 1.$

3. $\ldots\ldots = 1 - 2 (\sin A)^2.$

4. $\ldots\ldots = \dfrac{1 - (\tan A)^2}{1 + (\tan A)^2}.$

5. $\ldots\ldots = \dfrac{2 - (\sec A)^2}{(\sec A)^2}.$

48. In the same way, the following values of $\sin 2A$, and of $\cos 2A$, can be found in terms of $\cot A$, $\operatorname{cosec} A$, and $\operatorname{versin} A$.

$$\operatorname{Sin} 2A = \frac{2 \cot A}{1 + (\cot A)^2},$$

$$\sin 2A = \frac{2 \sqrt{\{(\operatorname{cosec} A)^2 - 1\}}}{(\operatorname{cosec} A)^2},$$

$$\sin 2A = 2 . (1 - \operatorname{versin} A) . \sqrt{\{2 \operatorname{vers} A - (\operatorname{vers} A)^2\}}.$$

$$\operatorname{Cos} 2A = \frac{(\cot A)^2 - 1}{(\cot A)^2 + 1},$$

$$\cos 2A = \frac{(\operatorname{cosec} A)^2 - 2}{(\operatorname{cosec} A)^2},$$

$$\cos 2A = 1 - 2 . \{2 \operatorname{vers} A - (\operatorname{vers} A)^2\}.$$

49. It may be remarked, that the easiest method of deducing formulæ such as these, is first to express $\sin 2A$ and $\cos 2A$ in terms of $\sin A$ and $\cos A$. Thus, let it be required to prove that

$$\cos 2A = \frac{(\operatorname{cosec} A)^2 - 2}{(\operatorname{cosec} A)^2}.$$

Since, $$\cos 2A = (\cos A)^2 - (\sin A)^2$$
$$= 1 - 2(\sin A)^2;$$

And, $$\sin A = \frac{1}{\operatorname{cosec} A}; \quad \text{Art. 27. (5).}$$

$$\therefore \cos 2A = 1 - \frac{2}{(\operatorname{cosec} A)^2} = \frac{(\operatorname{cosec} A)^2 - 2}{(\operatorname{cosec} A)^2}.$$

50. Since, $$\sin(A + B) = \sin A . \cos B + \cos A . \sin B,$$
and, $$\sin(A - B) = \sin A . \cos B - \cos A . \sin B;$$

$\therefore$ by adding and subtracting, we have

$$\sin(A + B) + \sin(A - B) = 2 \sin A . \cos B \ldots\ldots (1).$$
$$\sin(A + B) - \sin(A - B) = 2 \cos A . \sin B \ldots\ldots (2).$$

Similarly it may be shewn that,

$$\cos(A + B) + \cos(A - B) = 2 \cos A . \cos B \ldots\ldots (3).$$
$$\cos(A - B) - \cos(A + B) = 2 \sin A . \sin B \ldots\ldots (4).$$

51. *To find the values of* $\sin A \pm \sin B$, *and* $\cos A \pm \cos B$, *in terms of the sines and cosines of* $\frac{1}{2}(A + B)$, *and* $\frac{1}{2}(A - B)$.

Since, $$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B),$$
and, $$B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B);$$

$$\therefore \sin A = \sin\tfrac{1}{2}(A + B) . \cos\tfrac{1}{2}(A - B) + \cos\tfrac{1}{2}(A + B) . \sin\tfrac{1}{2}(A - B),$$
$$\sin B = \sin\tfrac{1}{2}(A + B) . \cos\tfrac{1}{2}(A - B) - \cos\tfrac{1}{2}(A + B) . \sin\tfrac{1}{2}(A - B);$$

$$\therefore\ \sin A + \sin B = 2 \sin \tfrac{1}{2}(A+B) \,.\, \cos \tfrac{1}{2}(A-B) \ldots\ldots (1).$$

$$\sin A - \sin B = 2 \cos \tfrac{1}{2}(A+B) \,.\, \sin \tfrac{1}{2}(A-B) \ldots\ldots (2).$$

Similarly,

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \,.\, \cos \tfrac{1}{2}(A-B) \ldots\ldots (3).$$

$$\cos B - \cos A = 2 \sin \tfrac{1}{2}(A+B) \,.\, \sin \tfrac{1}{2}(A-B) \ldots\ldots (4).$$

These four formulæ might have been deduced from the formulæ of the last Article, by making,

$$A + B = S, \quad \text{and} \quad A - B = D,$$

in which case,

$$A = \tfrac{1}{2}(S + D), \quad B = \tfrac{1}{2}(S - D).$$

52. The formulæ of the last two Articles are necessary to be remembered; the following are not of such frequent occurrence.

Dividing (2) by (1), Art. 51,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{1}{2}(A+B) \,.\, \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \,.\, \cos \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

Similarly, dividing (4) by (3),

$$\frac{\cos B - \cos A}{\cos A + \cos B} = \tan \tfrac{1}{2}(A+B) \,.\, \tan \tfrac{1}{2}(A-B).$$

So also,

$$\frac{\sin A \pm \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A \pm B);$$

$$\frac{\sin A \pm \sin B}{\cos B - \cos A} = \text{cotan}\, \tfrac{1}{2}(A \mp B).$$

53. $\text{Tan}\, A \pm \tan B = \dfrac{\sin A}{\cos A} \pm \dfrac{\sin B}{\cos B}$

$$= \frac{\sin A \,.\, \cos B \pm \cos A \,.\, \sin B}{\cos A \,.\, \cos B}$$

$$= \frac{\sin (A \pm B)}{\cos A \,.\, \cos B}.$$

Similarly, $\cot B \pm \cot A = \dfrac{\sin (A \pm B)}{\sin A \,.\, \sin B}.$

54. $\text{Sin}\,(A + B) \,.\, \sin (A - B)$

$$= (\sin A)^2 \,.\, (\cos B)^2 - (\cos A)^2 \,.\, (\sin B)^2$$

$$= (\sin A)^2 \,.\, \{1 - (\sin B)^2\} - \{1 - (\sin A)^2\} \,.\, (\sin B)^2$$

$$= (\sin A)^2 - (\sin B)^2.$$

Similarly, $\sin (A + B) \,.\, \sin (A - B) = (\cos B)^2 - (\cos A)^2.$

In like manner it may be shewn that

$$\cos (A + B) \,.\, \cos (A - B) = (\cos A)^2 - (\sin B)^2,$$

or, ... $= (\cos B)^2 - (\sin A)^2.$

55. *To prove that*

$$\left\{\begin{array}{l} \sin nA + \sin (n-2) A = 2 \sin (n-1) A \,.\, \cos A, \\ \cos nA + \cos (n-2) A = 2 \cos (n-1) A \,.\, \cos A. \end{array}\right.$$

$$\text{Sin}\, nA = \sin \{(n-1) \,.\, A + A\}$$

$$= \sin (n-1) A \,.\, \cos A + \cos (n-1) A \,.\, \sin A.$$

And, $\sin (n-2) A = \sin \{(n-1) A - A\}$

$$= \sin (n-1) A \,.\, \cos A - \cos (n-1) A \,.\, \sin A$$

$\therefore\ \sin nA + \sin (n-2) A = 2 \sin (n-1) A \,.\, \cos A$..................(1).

Again; in the same manner,

$$\cos nA = \cos (n-1) A . \cos A - \sin (n-1) A . \sin A,$$

And, $\cos (n-2) A = \cos (n-1) A . \cos A + \sin (n-1) A . \sin A$;

$$\therefore \cos nA + \cos (n-2) A = 2 \cos (n-1) A . \cos A \ldots\ldots\ldots (2).$$

Cor. If $n = 2$, we have from (1),

$$\sin 2A = 2 \sin A . \cos A,$$

.... from (2),

$$\cos 2A + 1 = 2 (\cos A)^2,$$

$$\text{or,} \quad \cos 2A = 2 (\cos A)^2 - 1.$$

If $n = 3$; from (1),

$$\begin{aligned} \sin 3A &= 2 \sin 2A . \cos A - \sin A \\ &= 4 \sin A . (\cos A)^2 - \sin A \\ &= 4 \sin A . \{1 - (\sin A)^2\} - \sin A \\ &= 3 \sin A - 4 (\sin A)^3, \end{aligned}$$

from (2),

$$\begin{aligned} \cos 3A &= 2 \cos 2A . \cos A - \cos A \\ &= 2 \cos A . \{2 (\cos A)^2 - 1\} - \cos A \\ &= 4 (\cos A)^3 - 3 \cos A; \end{aligned}$$

and by successive substitutions, $\sin 4A$, $\sin 5A \ldots$ $\cos 4A$, $\cos 5A \ldots$ might be found.

56. In like manner,

$$\sin nA - \sin (n-2) A = 2 \cos (n-1) A . \sin A \ldots\ldots (1),$$

$$\cos (n-2) A - \cos nA = 2 \sin (n-1) A . \sin A \ldots\ldots (2).$$

57. From the latter of these formulæ we may, by means of successive substitutions, find the cosine of nA in terms of the sines of A and its multiples.

Suppose n to be even, and $=2m$.

$$\therefore\ \cos 2(m-1)A - \cos 2mA = 2\sin(2m-1)A \,.\, \sin A,$$

so
$$\cos 2(m-2)A - \cos 2(m-1)A = 2\sin(2m-3)A \,.\, \sin A,$$

$$\cos 2(m-3)A - \cos 2(m-2)A = 2\sin(2m-5)A \,.\, \sin A,$$

$$\&c. \qquad = \qquad \&c.$$

$$\cos 2(m-m)A - \cos 2\{m-(m-1)\}A = 2\sin A \,.\, \sin A,$$

$\therefore$ by addition we have,

(since $\cos 2(m-m)A$, or $\cos 0^0$, $= 1$,)

$$1 - \cos 2mA = 2\sin A \,.\left\{\begin{matrix} \sin(2m-1)A + \sin(2m-3)A + \dots \\ \dots + \sin 3A + \sin A \end{matrix}\right\}.$$

$$\therefore\ \cos 2mA = 1 - 2\sin A \,.\left\{\begin{matrix} \sin(2m-1)A + \sin(2m-3)A + \dots \\ \dots + \sin 3A + \sin A \end{matrix}\right\} (1).$$

If n is odd, and $=2m+1$, we should have got,

$$\cos(2m+1)A = \cos A - 2\sin A \,.\left\{\begin{matrix} \sin 2mA + \sin 2(m-1)A + \dots \\ \dots + \sin 4A + \sin 2A \end{matrix}\right\} (2).$$

If $m=1$, these formulæ give us,

$$\cos 2A = 1 - 2\sin A \,.\, \sin A = 1 - 2(\sin A)^2.$$

$$\cos 3A = \cos A - 2\sin A \,.\, \sin 2A = \cos A - 2\sin A \,.\, 2\sin A \,.\, \cos A$$
$$= \cos A - 4\cos A \,.\, \{1 - (\cos A)^2\} = 4(\cos A)^3 - 3\cos A.$$

58. *To find the sines and cosines of* 18°, 72°, 36°, *and* 54°.

$$\text{Sin } 36^0 = \cos(90^0 - 36^0) = \cos 54^0,$$

$$\text{or, (if } 18^0 = A\text{)}, \quad \sin 2A = \cos 3A;$$

$$\therefore\ 2\sin A \,.\, \cos A = 2\cos 2A \,.\, \cos A - \cos A, \quad \text{Art. } 55;$$

$$\therefore\ 2\sin A = 2\cos 2A - 1 = 2 \,.\, \{1 - 2(\sin A)^2\} - 1;$$

$$\therefore\ 4(\sin A)^2 + 2\sin A = 1.$$

And, solving this equation, we have, $\sin A = \dfrac{\pm\sqrt{5} - 1}{4}$.

Where the positive sign is to be taken, because sin 18° is a positive quantity.

[There are other angles, satisfying the equation $\sin 2A = \cos 3A$, whose sines are $-\dfrac{\sqrt{5}+1}{4}$. Some of these angles will be determined in the next Article.]

$$\therefore\ \frac{\sqrt{5} - 1}{4} = \sin 18^0 = \cos(90^0 - 18^0) = \cos 72^0 \ \ldots\ldots\ldots\ldots (1).$$

$$\text{And } (\cos 18^0)^2 = 1 - (\sin 18^0)^2 = 1 - \frac{6 - 2\sqrt{5}}{16} = \frac{10 + 2\sqrt{5}}{16},$$

$$\therefore\ \cos 18^0 = \frac{\sqrt{(10 + 2\sqrt{5})}}{4} = \sin 72^0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\text{Again; } \sin 54^0 = \cos 36^0 = \cos 2 \times 18^0 = (\cos 18^0)^2 - (\sin 18^0)^2$$

$$= \frac{10 + 2\sqrt{5}}{16} - \frac{6 - 2\sqrt{5}}{16}$$

$$= \frac{1 + \sqrt{5}}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$(\cos 54^0)^2 = 1 - (\sin 54^0)^2 = 1 - \frac{6 + 2\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16},$$

$$\therefore\ \cos 54^0 = \frac{\sqrt{(10 - 2\sqrt{5})}}{4} = \sin 36^0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

59. *To find some of the values of* A *which satisfy the equation*, $\sin 2A = \cos 3A$.

$$\cos 3A = \sin 2A = \cos (90^0 - 2A)$$
$$= \cos \{4n . 90^0 \pm (90^0 - 2A)\}. \quad \text{Art. 25.}$$
$$= \cos \{(4n \pm 1) 90^0 \mp 2A\}.$$

Let $3A = (4n \pm 1) 90^0 \mp 2A$; $\therefore 3A \pm 2A = (4n \pm 1) 90^0$.

Taking the upper signs together,

$$A = (4n + 1) \frac{90^0}{5} = (4n + 1) 18^0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}.$$

Taking the lower signs together,

$$A = (4n - 1) 90^0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

[Again,

$$\cos 3A = \sin 2A = \cos (90^0 - 2A)$$
$$= - \cos \{(4n + 2) 90^0 \pm (90^0 - 2A)\}. \quad \text{Art. 25.}$$
$$= \cos \{180^0 \pm [(4n + 2) 90^0 \pm (90^0 - 2A)]\}. \quad \text{Art. 23.}$$

Let $3A = 180^0 \pm \{(4n + 2) 90^0 \pm (90^0 - 2A)\}$;

$$\therefore 3A \pm 2A = 180^0 \pm (4n + 2) 90^0 \pm 90^0.$$

And by solving this equation, we obtain other values of A which satisfy the proposed equation.]

For n writing 0, 1, 2, 3......successively, we have

from (1); $A = 18^0$; $= 5 \times 18^0$; $= 9 \times 18^0$; $= 13 \times 18^0$;...

from (2); $A = -90^0$; $= 3 \times 90^0$; $= 7 \times 90^0$; $= 11 \times 90^0$;...

Cor. The angle 13.18^0, or 234^0, is one of the angles alluded to in the last Article whose sines are $-\frac{\sqrt{5} + 1}{4}$.

For $\sin 234^0 = \sin (180^0 + 54^0) = - \sin 54^0$ Art. 22. (2).

$$= - \frac{1 + \sqrt{5}}{4}. \quad \text{Art. 58. (3).}$$

60. *If an angle receive any increment, to find the corresponding increment of the sine of the angle.*

Let A receive an increment δA, and let the corresponding increment of the sine of A be represented by $\Delta \sin A$.

Then, $\Delta \sin A = \sin(A + \delta A) - \sin A$

$$= \sin A . \cos \delta A + \cos A . \sin \delta A - \sin A$$

$$= \cos A . \sin \delta A - \sin A . (1 - \cos \delta A)$$

$$= \cos A . \sin \delta A - \sin A . 2\left(\sin \frac{\delta A}{2}\right)^2 \quad \text{Art. 39.}$$

$$= \cos A . \sin \delta A . \left\{1 - \tan A . \frac{2\left(\sin \frac{\delta A}{2}\right)^2}{\sin \delta A}\right\}$$

$$= \cos A . \sin \delta A . \left\{1 - \tan A . \frac{2\left(\sin \frac{\delta A}{2}\right)^2}{2 \sin \frac{\delta A}{2} . \cos \frac{\delta A}{2}}\right\}$$

$$= \cos A . \sin \delta A . \left\{1 - \tan A . \tan \frac{\delta A}{2}\right\}.$$

Cor. If δA be very small, $\tan \frac{\delta A}{2}$ is very small. In this case, if $\tan A$ be not exceedingly large, (that is, if A be not nearly equal to $\overline{2n+1} . 90^\circ$, n being an integer), $\tan A . \tan \frac{\delta A}{2}$ is a very small quantity, and may be neglected in comparison with unity. We have therefore, *when these two conditions are fulfilled*,—and not otherwise,—

$$\Delta \sin A = \cos A . \sin \delta A.$$

Note. Hence it appears that if A be any angle of a triangle, this result cannot be applied in any particular case to determine the corresponding increment of the sine of the angle, when the angle itself has received a given *small* increment, if the angle be nearly equal to a right angle.

61. *If an angle receive any increment, to find the corresponding decrement of the cosine of the angle.*

$$\begin{aligned}\Delta \cos A &= \cos(A + \delta A) - \cos A \\ &= \cos A \,.\, \cos \delta A - \sin A \,.\, \sin \delta A - \cos A \\ &= -\sin A \,.\, \sin \delta A - \cos A \,.\, (1 - \cos \delta A) \\ &= -\sin A \,.\, \sin \delta A \,.\, \left\{1 + \cot A \,.\, \frac{2\left(\sin \frac{\delta A}{2}\right)^2}{\sin \delta A}\right\} \\ &= -\sin A \,.\, \sin \delta A \,.\, \left\{1 + \cot A \,.\, \tan \frac{\delta A}{2}\right\}.\end{aligned}$$

COR. 1. As in the last Article,—if δA be very small, and $\cot A$ be not at the same time exceedingly large, (*i. e.* if A be not $= 2n \,.\, 90^0$ nearly), $\cot A \,.\, \tan \frac{\delta A}{2}$ may be neglected with respect to unity; and we then have,

$$\Delta \cos A = -\sin A \,.\, \sin \delta A.$$

NOTE. It must be carefully borne in mind, that unless, (1), δA be a very small angle, and, (2), A be, besides, an angle which is not nearly equal to 0^0 or 180^0, this result cannot be applied to any particular case where A is an angle of a triangle.

COR. 2. If A be less than 90^0, $\cos A$ is positive, and $\sin A$ being also positive, in this case $\Delta \cos A$ is necessarily negative. That is, in angles less than a right angle, as the angle increases, its cosine decreases.

If A be greater than one right angle, but less than two, $\cos A$ is negative, and $\sin A$ being positive, $\Delta \cos A$ is negative. That is, when the angle is greater than one right angle, but less than two, as the angle increases, the cosine, being negative, also increases in magnitude.

62. *If an angle receive any increment, to find the corresponding increment of the secant of the angle.*

$$\Delta \sec A = \sec(A+\delta A) - \sec A$$

$$= \frac{1}{\cos(A+\delta A)} - \frac{1}{\cos A}$$

$$= \frac{\cos A - \cos(A+\delta A)}{\cos A \,.\, \cos(A+\delta A)}$$

$$= \frac{\sin A \,.\, \sin \delta A \,.\, \left\{1 + \cot A \,.\, \tan \dfrac{\delta A}{2}\right\}}{\cos A \,.\, (\cos A \,.\, \cos \delta A - \sin A \,.\, \sin \delta A)} \qquad \begin{array}{l}\text{Art. 61.}\\ \text{Art. 34.}\end{array}$$

$$= \frac{\sin A}{\cos A} \,.\, \frac{1}{\cos A} \,.\, \frac{\sin \delta A}{\cos \delta A} \,.\, \frac{1 + \cot A \,.\, \tan \dfrac{\delta A}{2}}{1 - \tan A \,.\, \tan \delta A}$$

$$= \tan A \,.\, \sec A \,.\, \tan \delta A \,.\, \frac{1 + \cot A \,.\, \tan \dfrac{\delta A}{2}}{1 - \tan A \,.\, \tan \delta A}.$$

Cor. If δA be very small, and neither $\tan A$ nor $\cot A$ be very large, (that is, if A be not nearly equal to $n\,.\,90^0$), $\cot A \,.\, \tan \dfrac{\delta A}{2}$ and $\tan A \,.\, \tan A\, \delta A$ will both be very small, and may be neglected when compared with unity. In this case therefore we have

$$\Delta \sec A = \tan A \,.\, \sec A \,.\, \tan \delta A.$$

Note. It is to be remarked, that before this result can be applied in any case where A is an angle of a triangle, we must have δA a very small angle, and moreover, A must not be nearly equal to 0, 90^0, or 180^0.

63. *If an angle receive any increment, to find the corresponding increment of the tangent of the angle.*

$$\Delta \tan A = \tan(A + \delta A) - \tan A$$

$$= \frac{\sin(A+\delta A)}{\cos(A+\delta A)} - \frac{\sin A}{\cos A}$$

$$= \frac{\sin(A+\delta A) . \cos A - \cos(A+\delta A) . \sin A}{\cos A . (\cos A . \cos \delta A - \sin A . \sin \delta A)}.$$

But $\sin(A+\delta A) . \cos A - \cos(A+\delta A) . \sin A$

$$= \sin\{(A+\delta A) - A\} = \sin \delta A;$$

$$\therefore \Delta \tan A = \frac{\sin \delta A}{(\cos A)^2 . \cos \delta A . (1 - \tan A . \tan \delta A)}$$

$$= (\sec A)^2 . \tan \delta A . \frac{1}{1 - \tan A . \tan \delta A}.$$

Cor. If δA be very small, and *also* $\tan A$ be not very large,—that is, *if* A *be not* (2n + 1) 90° *nearly*,—we have

$$\Delta \tan A = (\sec A)^2 . \tan \delta A.$$

Note. If, therefore, A be an angle of a triangle, this result is not true when A is nearly equal to a right angle.

64. *For a given small increment of* A, *the increment of the sine is* $\gtreqless$ *the decrement of the cosine corresponding to the same increment of the angle, according as* cos A *is* $\gtreqless$ sin A; A *not being very small, or nearly a multiple of* 90°.

For $\Delta \sin A = \cos A . \sin \delta A$, if A be not nearly $(2n+1) 90^\circ$.
Art. 60. Cor.

$\Delta \cos A = - \sin A . \sin \delta A$, if A be not nearly $2n . 90^\circ$.
Art. 61. Cor. 1.

Hence, if A be not very small, or nearly a multiple of 90^0,

$$\Delta \sin A \text{ is } \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} (-\Delta \cos A), \text{ as } \cos A \text{ is } \begin{smallmatrix} > \\ = \\ < \end{smallmatrix} \sin A.$$

Cor. In angles less than 90^0, $\Delta \sin A$ is $\begin{smallmatrix} > \\ < \end{smallmatrix}$ $(-\Delta \cos A)$

as A is $\begin{smallmatrix} < \\ > \end{smallmatrix}$ 45^0. Art. 31.

65. Def. By $\tan^{-1} t$ is meant the angle whose tangent is t; *i. e.* if $t = \tan A$, then $A = \tan^{-1} t$.

So, by $\sin^{-1} s$ and $\cos^{-1} c$, &c. are meant respectively the angle whose sine is s, and that whose cosine is c, &c.

66. *To shew that* $\tan^{-1} t_1 + \tan^{-1} t_2 = \tan^{-1} \dfrac{t_1 + t_2}{1 - t_1 t_2}$, *and* $\tan^{-1} t_1 - \tan^{-1} t_2 = \tan^{-1} \dfrac{t_1 - t_2}{1 + t_1 t_2}$.

Let $\tan A = t_1$, and $\tan B = t_2$.

Then, by definition, $A = \tan^{-1} t_1$, and $B = \tan^{-1} t_2$.

$$\text{Now} \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B};$$

$$\therefore \text{ by def.} \quad A + B = \tan^{-1} \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B};$$

$$\text{Or,} \quad \tan^{-1} t_1 + \tan^{-1} t_2 = \tan^{-1} \frac{t_1 + t_2}{1 - t_1 t_2} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Similarly it may be shewn that

$$\tan^{-1} t_1 - \tan^{-1} t_2 = \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

67. *If* $t_1, t_2, \ldots\ldots t_n$, *be the tangents of any angles, then* $\tan^{-1}t_1 - \tan^{-1}t_n$

$$= \tan^{-1}\frac{t_1 - t_2}{1 + t_1t_2} + \tan^{-1}\frac{t_2 - t_3}{1 + t_2t_3} + \ldots + \tan^{-1}\frac{t_{n-1} - t_n}{1 + t_{n-1}t_n}.$$

For,
$$\tan^{-1}t_1 - \tan^{-1}t_2 = \tan^{-1}\frac{t_1 - t_2}{1 + t_1t_2}$$

$$\tan^{-1}t_2 - \tan^{-1}t_3 = \tan^{-1}\frac{t_2 - t_3}{1 + t_2t_3},$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$\tan^{-1}t_{n-1} - \tan^{-1}t_n = \tan^{-1}\frac{t_{n-1} - t_n}{1 + t_{n-1}t_n};$$

∴ by addition; $\tan^{-1}t_1 - \tan^{-1}t_n$

$$= \tan^{-1}\frac{t_1 - t_2}{1 + t_1t_2} + \tan^{-1}\frac{t_2 - t_3}{1 + t_2t_3} + \ldots + \tan^{-1}\frac{t_{n-1} - t_n}{1 + t_{n-1}t_n}.$$

68. Examples of questions which may be solved by the application of formulæ proved in this, and the preceding, Chapter.

(1). *To prove that* $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A.$

[It is here required to bring the proposed fraction into one whose denominator shall be $\cos 2A$, or $(\cos A)^2 - (\sin A)^2$. Multiply, therefore, the numerator and denominator by the numerator, and we have,]

$$\frac{\cos A+\sin A}{\cos A-\sin A} = \frac{(\cos A+\sin A)^2}{(\cos A)^2-(\sin A)^2} = \frac{(\cos A)^2+(\sin A)^2+2\cos A\,.\sin A}{(\cos A)^2-(\sin A)^2}$$

$$= \frac{1 + \sin 2A}{\cos 2A}; \text{ by Art. 27. (6), and Arts. 38, 39.}$$

$$= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A} = \sec 2A + \tan 2A.$$

(2). *To prove that* $\cos 2\mathrm{A} = \dfrac{1}{1 + \tan 2\mathrm{A} \,.\, \tan \mathrm{A}}$.

[The equation becomes, when inverted,

$$\frac{1}{\cos 2A} = 1 + \tan 2A \,.\, \tan A;$$

where we perceive, that if the right hand side were expressed in terms of the sines and cosines of A and $2A$, and then put into a fractional form,' the denominator would be $\cos 2A \,.\, \cos A$, and the numerator would involve the sines and cosines of the same angles. Our first object therefore will be to express $\dfrac{1}{\cos 2A}$ in a fraction of such a form.]

$$\text{Now, } \frac{1}{\cos 2A} = \frac{\cos A}{\cos 2A \,.\, \cos A} = \frac{\cos (2A - A)}{\cos 2A \,.\, \cos A}$$

$$= \frac{\cos 2A \,.\, \cos A + \sin 2A \,.\, \sin A}{\cos 2A \,.\, \cos A}$$

$$= 1 + \frac{\sin 2A}{\cos 2A} \,.\, \frac{\sin A}{\cos A} = 1 + \tan 2A \,.\, \tan A;$$

$$\therefore \cos 2A = \frac{1}{1 + \tan 2A \,.\, \tan A}.$$

Note. In the following Examples it is required to determine an angle from some given relation existing between its goniometrical functions and those of either a multiple of the angle sought, or of some given angle;—and conversely.

(3). *Determine* A *from the equation*, $\sin 2\mathrm{A} = \sin \mathrm{A}$.

$\operatorname{Sin} A = \sin 2A = 2 \sin A \,.\, \cos A$. Art. 38.

$\therefore 2 \cos A = 1$, and $\cos A = \frac{1}{2}$. Wherefore $A = 60^{0}$. Art. 32.

(4). *Determine* B *from the equation,*

$$\sin A + \sin(2B + A) - \sin(2B - A) = \sin(B + A) - \sin(B - A);$$

$$\therefore \sin A + 2\cos 2B . \sin A = 2\cos B . \sin A; \text{ Art. 50, (2).}$$

$$\therefore 1 + 2\cos 2B = 2\cos B;$$

$$\therefore 1 + 2 . \{2(\cos B)^2 - 1\} = 2\cos B. \quad \text{Art. 39.}$$

Whence we get,

$$\cos B = \frac{1 \pm \sqrt{5}}{4}.$$

Now, Art. 58. (3), $\dfrac{1 + \sqrt{5}}{4} = \cos 36^0.$

And, Art. 58. (1), $\dfrac{\sqrt{5} - 1}{4} = \cos 72^0;$

$$\therefore \frac{1 - \sqrt{5}}{4} = -\frac{\sqrt{5} - 1}{4} = -\cos 72^0 = \cos(180^0 - 72^0) = \cos 108^0.$$

Wherefore 36^0, and 108^0, are the values of B required.

(5). *If* $\text{m} . \dfrac{\tan(\text{A} - \text{B}}{(\cos \text{B})^2} = \text{n} . \dfrac{\tan \text{B}}{\{\cos(\text{A} - \text{B})\}^2}$, *required to prove that,* $\tan(\text{A} - 2\text{B}) = \dfrac{\text{n} - \text{m}}{\text{n} + \text{m}} . \tan \text{A}.$

$$\frac{n}{m} = \frac{\tan(A - B) . \{\cos(A - B)\}^2}{\tan B . (\cos B)^2} = \frac{\sin(A - B) . \cos(A - B)}{\sin B . \cos B}$$

$$= \frac{2 . \sin(A - B) . \cos(A - B)}{2\sin B . \cos B} = \frac{\sin 2(A - B)}{\sin 2B}.$$

Whence we obtain,

$$\frac{n-m}{n+m}=\frac{\sin 2(A-B)-\sin 2B}{\sin 2(A-B)+\sin 2B}$$

$$=\frac{2 \,.\cos A \,.\sin (A-2B)}{2 \,.\sin A \,.\cos (A-2B)} \quad \text{Art. 50, (2), (1),}$$

$$=\frac{\tan (A-2B)}{\tan A};$$

$$\therefore\ \tan (A-2B)=\frac{n-m}{n+m}.\tan A.$$

(6). *To prove that*, $2\cos 11^0, 15'=\sqrt{2+\sqrt{\{2+\sqrt{2}\}}}$.

$$\operatorname{Cos} 45^0=\frac{1}{\sqrt{2}};$$

$$\therefore\ 2\cos 45^0=\sqrt{2},$$

$$\therefore\ 2\,.\left\{2\left(\cos\frac{45^0}{2}\right)^2-1\right\}=\sqrt{2};$$

$$\therefore\ 4\left(\cos\frac{45^0}{2}\right)^2=2+\sqrt{2};$$

$$\therefore\ 2\cos\frac{45^0}{2}=\sqrt{2+\sqrt{2}};$$

$$\therefore\ 2\,.\left\{2\left(\cos\frac{45^0}{2^2}\right)^2-1\right\}=\sqrt{2+\sqrt{2}};$$

$$\therefore\ 4\left(\cos\frac{45^0}{2^2}\right)=2+\sqrt{2+\sqrt{2}};$$

$$\therefore\ 2\cos\frac{45^0}{2^2},\ \text{or}\ 2\cos 11^0, 15', =\sqrt{2+\sqrt{\{2+\sqrt{2}\}}}.$$

Cor. By repeating the same process n times, it would appear in like manner, that

$$2\cos\frac{45^0}{2^n} = \sqrt{2+\sqrt{(2+\&c.)}}.$$

Where 2, with the sign of a square root over it, appears $n+1$ times in the second member of the equation,—the square root every time reaching to the end of the expression.

(7). *If* $x.\tan A = (\sqrt{1+x}-1).(\sqrt{1-x}+1)$, *required to prove that* $x = 2\cos 2A$.

$$x.\tan A = (\sqrt{1+x}-1).(\sqrt{1-x}+1).\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = x.\frac{\sqrt{1-x}+1}{\sqrt{1+x}+1};$$

$$\therefore\ \sqrt{1-x}+1 = (\sqrt{1+x}+1).\tan A;$$

$$\therefore\ (1-\tan A)^2 = \{\sqrt{1+x}.\tan A - \sqrt{1-x}\}^2;$$

$$\therefore\ 1 - 2\tan A + (\tan A)^2 = 1 + (\tan A)^2 - x.\{1-(\tan A)^2\}$$
$$- 2\sqrt{1-x^2}.\tan A;$$

$$\therefore\ 2\tan A = x.\{1-(\tan A)^2\} + 2\sqrt{1-x^2}.\tan A;$$

$$\therefore\ 1 = x.\frac{1-(\tan A)^2}{2\tan A} - \sqrt{1-x^2},$$

and $\dfrac{1-(\tan A)^2}{2\tan A} = \dfrac{1}{\tan 2A}$ (Art. 47.) $= \cot 2A$;

$$\therefore\ \sqrt{1-x^2} = 1 - x.\cot 2A;$$

$$\therefore\ 1 - x^2 = 1 - 2x.\cot 2A + x^2.(\cot 2A)^2;$$

$$\therefore\ x^2.\{1+(\cot 2A)^2\} - 2x.\cot 2A = 0;$$

$$\therefore\ x = 0;\quad \text{or}\ x = \frac{2\cot 2A}{1+(\cot 2A)^2} = 2.\frac{\cot 2A}{(\operatorname{cosec} 2A)^2} = 2\cos 2A.$$

(8). *If* $\tan A \,.\, \tan B = \{\tan(A+B)\}^2 - \{\tan(A-B)\}^2$, *to find* $\cos B$.

$$\tan A \,.\, \tan B = \{\tan(A+B) + \tan(A-B)\}\,.\,\{\tan(A+B) - \tan(A-B)\}$$

$$= \frac{\sin 2A \,.\, \sin 2B}{\{\cos(A+B) \,.\, \cos(A-B)\}^2}; \quad \text{Art. 53.}$$

$$\therefore \frac{\sin A \,.\, \sin B}{\cos A \,.\, \cos B} = \frac{2\sin A \,.\, \cos A \,.\, 2\sin B \,.\, \cos B}{\{\cos(A+B) \,.\, \cos(A-B)\}^2};$$

$$\therefore 4 \,.\, (\cos A)^2 \,.\, (\cos B)^2 = \{\cos(A+B) \,.\, \cos(A-B)\}^2;$$

$$\therefore 2\cos A \,.\, \cos B = \cos(A+B) \,.\, \cos(A-B)$$

$$= (\cos B)^2 - (\sin A)^2; \quad \text{Art. 54.}$$

$$\therefore (\cos B)^2 - 2\cos A \,.\, \cos B + (\cos A)^2 = (\sin A)^2 + (\cos A)^2 = 1;$$

$$\therefore \cos B - \cos A = \pm 1;$$

$$\therefore \cos B = \cos A \pm 1.$$

Taking the positive sign,

$$\cos B = 1 + \cos A = 2\left(\cos\frac{A}{2}\right)^2 \ldots\ldots\ldots\ldots\ldots(1).$$

Taking the negative sign,

$$\cos B = \cos A - 1 = -2\left(\sin\frac{A}{2}\right)^2;$$

$$\therefore \cos(180^0 - B) = -\cos B = 2\left(\sin\frac{A}{2}\right)^2 \ldots\ldots\ldots\ldots(2).$$

Note. In the Corollaries to Arts. 37 and 45, the relation has been found which exists between the goniometrical functions of three angles A, B, C, which are connected together by the condition, $A + B + C = 180^0$; the following question, conversely, requires the relation between three angles, when equations are given connecting their goniometrical functions together.

(9). *If* $\left\{\begin{array}{l}\tan \mathrm{A} = (\tan \mathrm{C})^3\\ \tan 2\mathrm{C} = 2\tan 2\mathrm{B}\end{array}\right\}$, *then* A, B, C *are angles in arithmetical progression.*

[We here want to prove that $2B = A + C$.]

Now, $\tan 2B = \frac{1}{2}\tan 2C = \frac{\tan C}{1-(\tan C)^2}$ Art. 47.

$$= \frac{\tan C}{1 - \frac{\tan A}{\tan C}},\quad \left(\text{since } (\tan C)^2 = \frac{\tan A}{\tan C}, \text{ by the first equation,}\right)$$

$$= \frac{(\tan C)^2}{\tan C - \tan A} = \frac{(\tan C + \tan A)\,.\,(\tan C)^2}{(\tan C)^2 - (\tan A)^2}$$

$$= \frac{\tan C + \tan A}{1 - \left(\frac{\tan A}{\tan C}\right)^2};$$

and $\left(\frac{\tan A}{\tan C}\right)^2 = \tan A\,.\,\frac{\tan A}{(\tan C)^3} = \tan A\,.\,\tan C$, by the first equation,

$$\therefore\ \tan 2B = \frac{\tan A + \tan C}{1 - \tan A\,.\,\tan C} = \tan(A + C);\quad \text{Art. 43.}$$

$\therefore\ 2B = A + C$, and $B - A = C - B$; or A, B, C are in arithmetical progression.

(10). *To prove that*

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = 45^0.$$

$$\mathrm{Tan}^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{1}{3}+\frac{1}{5}}{1 - \frac{1}{3}\cdot\frac{1}{5}},\quad \text{Art. 66,}\quad = \tan^{-1}\frac{4}{7},$$

So $$\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}=\tan^{-1}\frac{3}{11};$$

$$\therefore\quad \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}$$

$$=\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{3}{11}=\tan^{-1}\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\cdot\frac{3}{11}}$$

$$=\tan^{-1}\frac{65}{65}=\tan^{-1}1.$$

And the angle whose tangent is 1, is half a right angle, or 45^0;

$$\therefore\ \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=45^0.$$

The Appendices I, II, III, on the Logarithmic Tables of Numbers and of Goniometrical Functions, ought to be read before entering on the next Chapter.

CHAPTER IV.

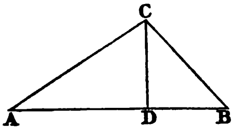
ON THE SOLUTION OF TRIANGLES.

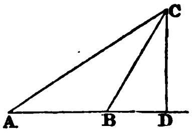
69. A TRIANGLE consists of six parts, namely, three sides and three angles; when three of these parts are given, (except they be the three angles), it will be shewn that the other three can, generally, be determined.

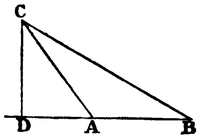
The number of degrees in the angles of a triangle will be designated by the letters A, B, C, which are placed at the angular points of the triangle, and the lengths of the sides respectively opposite to the angles A, B, C, by the letters a, b, c.

70. *The sines of the angles of a triangle are proportional to the sides respectively opposite.*

Let ABC be the triangle. Draw CD perpendicular to AB, or AB produced either way.







$$\text{Then } \sin CAB = \sin CAD = \frac{DC}{CA}.$$

(With reference to the third figure, see Art. 26.)

$$\text{Also, } \quad \sin CBA = \sin CBD = \frac{DC}{CB};$$

$$\therefore \quad \frac{\sin CAB}{\sin CBA} = \frac{CB}{CA};$$

$$\therefore \quad \frac{\sin A}{\sin B} = \frac{a}{b}; \quad \text{or,} \quad \frac{\sin A}{a} = \frac{\sin B}{b}.$$

In like manner, if a perpendicular were let fall from B upon the side opposite to it, it might be proved that,

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

Wherefore,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c};$$

the magnitudes of the lines a, b, c, being represented by the *number* of units of length they respectively contain; for otherwise $\sin A$ and a would not be quantities of the same kind, and consequently no ratio could exist between them.

71. Since (Euclid I. 32), the interior angles of a triangle are together equal to two right angles, we have

$$A + B + C = 180^0.$$

$$\text{Also;}\quad \frac{\sin A}{\sin B} = \frac{a}{b},$$

$$\frac{\sin B}{\sin C} = \frac{b}{c}.$$

And if three of the parts of the triangle be given, the remaining three parts may be determined by these three equations.

It is necessary however that one, at least, of the given parts be a side, or we have merely the *ratios* given which a, b, c bear to each other, and their *magnitudes* cannot be determined, because there are but *two* equations,

$$\text{namely,}\quad \frac{\sin A}{\sin B} = \frac{a}{c},\quad \text{and}\quad \frac{\sin B}{\sin C} = \frac{b}{c},$$

for determining the *three* unknown quantities a, b, c.

This also appears from the consideration that an indefinite number of equiangular triangles, of all possible degrees of magnitude, can be formed by drawing lines parallel to the sides of a given triangle.

72. There is however one case, commonly called "*the ambiguous case*," in which the equations of the last article are not sufficient to determine the triangle when three of the parts are given.

If two sides be given, and an angle opposite to one of them (a, b, A), *the triangle can be determined only in the case where the side opposite to the given angle is the greater of the two sides which are given;* i. e. *when* a *is greater than* b.

The equations of the last article become

$$B + C = 180^0 - A \text{ .... (i)},$$

$$\operatorname{Sin} B = \frac{b}{a} . \sin A \text{ .... (ii)},$$

$$c = b . \frac{\sin C}{\sin B} \text{ .... (iii)}.$$

If we can determine B from (ii), C and c are known from (i) and (iii), and the triangle is determined. Now the sine of an angle is equal to the sine of its supplement, and therefore there are *two* angles which satisfy (ii), the one greater and the other less than 90^0.

(1). Let a be greater than b;

$$\therefore \quad A > B. \quad \text{Eucl. I. 18.}$$

Now B cannot be greater than 90^0, for in that case $A + B$ would be greater than 180^0, which is impossible, (Eucl. I. 17.);

$$\therefore \quad B < 90^0,$$

and the *lesser* angle which satisfies (ii) is to be taken for the value of B.

(2). Let a be less than b;

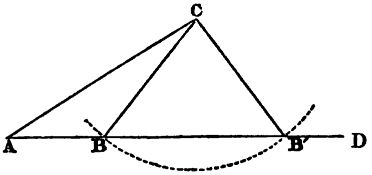
$$\therefore \quad A < B.$$

In this case there is nothing to guide us in taking the value of B from (ii), since the only limits,

$$B + A < 180^0, \quad \text{and} \quad B > A,$$

may be fulfilled whether B be greater or less than 90^0.

Thus if $CB = CB'$, it is evident that *both* the triangles ABC, $AB'C$ have a, b, A of the same values; also, in this case,



$\angle A$ is less than the exterior angle CBB', or the exterior angle $CB'D$;

i. e. A is less than $CB'A$, or CBA;

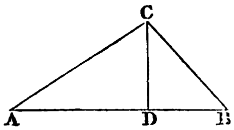
$$\therefore\ a < b.$$

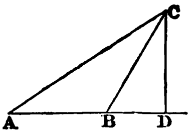
Which agrees with what has just been asserted.

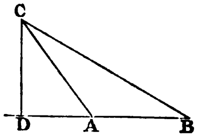
73. In the solution of triangles there are various artifices used, which will be pointed out in each case. We shall now prove some formulæ of great use in the solution of plane rectilineal figures.

74. *To find the cosine of an angle of a triangle in terms of the sides.*

Let ABC be a triangle, and from C draw CD perpendicular to AB, or AB produced either way.







Then, figs. 1, 2, $CB^2 = AC^2 + AB^2 - 2AB \cdot AD$, (Eucl. II. 13.)

fig. 3, $CB^2 = AC^2 + AB^2 + 2AB \cdot AD$, (Eucl. II. 12.)

$$\text{And}\quad \frac{AD}{AC} = \cos CAD,$$

$$= \cos CAB \text{ in figs. } 1,\ 2.$$

$$= -\cos CAB \text{ in fig. } 3. \qquad \text{Art. 26.}$$

Therefore, in each of these cases, we have

$$CB^2 = AC^2 + AB^2 - 2AB.AC.\cos CAB;$$

$$\therefore\ \cos CAB = \frac{AC^2 + AB^2 - CB^2}{2AB.AC}.$$

Or $\quad \text{Cos}\, A = \dfrac{b^2 + c^2 - a^2}{2bc}$

75. *To shew that* $\cos\dfrac{A}{2} = \sqrt{\dfrac{S.(S-a)}{bc}}$,

where $S = \dfrac{a+b+c}{2}$.

For $\quad 1 + \cos A = 1 + \dfrac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{b^2 + 2bc + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a).(b+c-a)}{2bc};$$

Now, $\quad 1 + \cos A = 2\left(\cos\dfrac{A}{2}\right)^2$, Art. 39. (2).

And $\quad S - a = \frac{1}{2}(a+b+c) - a = \frac{1}{2}(b+c-a)$;

$$\therefore\ 2\left(\cos\frac{A}{2}\right)^2 = \frac{(b+c+a).(b+c-a)}{2bc}$$

$$= \frac{2S.2(S-a)}{2bc};$$

$$\therefore\ \text{Cos}\frac{A}{2} = \sqrt{\frac{S.(S-a)}{bc}};$$

76. *To shew that* $\sin\frac{A}{2} = \sqrt{\frac{(S-b).(S-c)}{bc}}$.

By Art. 39. (3), we have

$$2\left(\sin\frac{A}{2}\right)^2 = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a+c-b).(a+b-c)}{2bc}.$$

Now $S - b = \frac{1}{2}(a+b+c) - b = \frac{1}{2}(a+c-b)$,

and $S - c = \frac{1}{2}(a+b-c)$;

$$\therefore\ 2\left(\sin\frac{A}{2}\right)^2 = \frac{2(S-b).2(S-c)}{2bc};$$

$$\therefore\ \text{Sin}\frac{A}{2} = \sqrt{\frac{(S-b).(S-c)}{bc}}.$$

The *positive* sign of the root must be taken, because A, being an angle of a triangle, is less than 180⁰, and therefore $\sin\frac{A}{2}$ is a positive quantity.

Cor. 1. $(\text{Sin}\,A)^2$

$$= 1 - (\cos A)^2 = (1+\cos A)\ (1-\cos A)$$

$$= \left\{1 + \frac{b^2+c^2-a^2}{2bc}\right\}.\left\{1 - \frac{b^2+c^2-a^2}{2bc}\right\}$$

$$= \frac{1}{4b^2c^2}.(b+c-a).(b+c+a).(a+b-c).(a+c-b)$$

$$= \frac{1}{4b^2c^2}.2S.2(S-a).2(S-b).2(S-c)$$

$$= \frac{4}{b^2c^2}.S.(S-a).(S-b).(S-c);$$

$$\therefore\ \text{Sin}\,A = \frac{2}{bc}\sqrt{\{S.(S-a).(S-b).(S-c)\}}.$$

Cor. 2. $$\text{Tan}\,\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(S-b).(S-c)}{S.(S-a)}}.$$

77. We will now explain the meaning of the double sign in the second member of the equation

$$\text{Cos}\,\frac{A}{2} = \pm\sqrt{\frac{S.(S-a)}{bc}}.$$

$$\text{Since}\quad \cos(4n.90^0 \pm A) = \cos A, \quad \text{Art. 25.}$$

$$\therefore\ \cos(4n.90^0 \pm A) = \frac{b^2+c^2-a^2}{2bc}.$$

Whence we get, in the same manner as in Art. 75,

$$\cos\left(2n.90^0 \pm \frac{A}{2}\right) = \pm\sqrt{\frac{S.(S-a)}{bc}}.$$

Now the first member of this equation ought to furnish us with two sets of angles whose cosines are of the same magnitude but of different signs.

First. Let n be even, and $= 2m$.

$$\text{Then Cos}\left(2n.90^0 \pm \frac{A}{2}\right) = \cos\left(4m.90^0 \pm \frac{A}{2}\right) = \cos\left(m.360^0 \pm \frac{A}{2}\right),$$

which, by making m equal to 0, 1, 2, 3,... successively, gives the series of angles

$$\pm\frac{A}{2}, \quad 360^0 \pm \frac{A}{2}, \quad 2\times 360^0 \pm \frac{A}{2}, \quad 3\times 360^0 \pm \frac{A}{2}, \ldots\ldots$$

whose cosines are all of the same magnitude as the cosine of $\pm\frac{A}{2}$, or of $+\frac{A}{2}$.

Second. Let n be odd, and $= 2m + 1$.

Then $\cos\left(2n \cdot 90^0 \pm \frac{A}{2}\right) = \cos\left\{4m \cdot 90^0 + \left(180^0 \pm \frac{A}{2}\right)\right\}$,

which gives the series of angles

$$180^0 \pm \frac{A}{2}, \quad 360^0 + \left(180^0 \pm \frac{A}{2}\right), \quad 2 \cdot 360^0 + \left(180^0 \pm \frac{A}{2}\right), \ldots\ldots$$

whose cosines are all of the same magnitude as the cosine of

$180^0 \pm \frac{A}{2}$, which $= \cos\left(180^0 - \frac{A}{2}\right) = -\cos\frac{A}{2}$.

Wherefore the first member of the equation *does* furnish us with two sets of angles whose cosines are of the same magnitudes but of different signs.

Cor. A, being an angle of a triangle, is less than 180^0;

$$\therefore \quad \operatorname{Cos}\frac{A}{2} = +\sqrt{\frac{S \cdot (S-a)}{bc}},$$

$$\text{And} \quad \operatorname{Cos}\left(180^0 - \frac{A}{2}\right) = -\sqrt{\frac{S \cdot (S-a)}{bc}}.$$

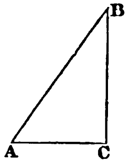
78. By a similar process of reasoning, sets of angles may be determined, which correspond to the double signs with which the expressions we have obtained as the values of $\sin\frac{A}{2}$, $\tan\frac{A}{2}$, and $\sin A$, are affected.

We now proceed to the solution of triangles.

I. ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

79. *The right angle, a side, and another part being given, to determine the remaining parts.*

Let ABC be a right-angled triangle, C being the right angle.



(1). *Let* c *and* A *be the other given quantities.*

$$\text{Then } \frac{b}{c} = \cos A; \quad \text{and } \frac{a}{c} = \sin A;$$

$$\therefore \begin{cases} \mathrm{l}_{10} b = \mathrm{l}_{10} c + L \cos A - 10; & \text{which determines } b, \\ \mathrm{l}_{10} a = \mathrm{l}_{10} c + L \sin A - 10; & \text{which determines } a. \end{cases}$$

Also, $\angle B = 90^0 - A$; which determines B.

Had B been the given angle, the method of solution for determining A would have been the same.

(2). *Let* A *and* b *be given.*

$$\text{Then } \frac{c}{b} = \sec A; \quad \text{and } \frac{a}{b} = \tan A;$$

$$\therefore \begin{cases} \mathrm{l}_{10} c = \mathrm{l}_{10} b + L \sec A - 10; & \text{which determines } c. \\ \mathrm{l}_{10} a = \mathrm{l}_{10} b + L \tan A - 10; & \text{which determines } a. \end{cases}$$

Also, $B = 90^0 - A$; which determines B.

(3). *Let* A *and* a *be given.*

$$\text{Then } \frac{a}{b} = \tan A; \quad \therefore b = \frac{a}{\tan A}, \quad \text{or } = a \,.\, \cot A.$$

$$\text{And } \mathrm{l}_{10} b = \mathrm{l}_{10} a - L \tan A + 10,$$

$$\text{or } = \mathrm{l}_{10} a + L \cot A - 10.$$

Again, $\frac{a}{c} = \sin A$;

$$\therefore \quad c = \frac{a}{\sin A}, \quad \text{or} = a \, . \operatorname{cosec} A.$$

And $\quad l_{10} c = l_{10} a - L \sin A + 10,$

$\qquad$ or $\quad = l_{10} a + L \operatorname{cosec} A - 10.$

Again, $\quad B = 90^{\circ} - A.$

(4). *Let* a, b *be given.*

$$\text{Then,} \quad \tan A = \frac{a}{b};$$

$$\therefore \quad L \tan A = l_{10} a - l_{10} b + 10.$$

Again, $\quad B = 90^{\circ} - A.$

Again, $\quad \frac{c}{b} = \sec A$;

$$\therefore \quad l_{10} c = l_{10} b + L \sec A - 10.$$

We might determine c from $c = \sqrt{(a^2 + b^2)}$, but the operation is tedious, particularly if a and b be large.

(5). *Let* c *and* a *be given.*

$$\text{Then,} \quad \sin A = \frac{a}{c};$$

$$\therefore \quad L \sin A = l_{10} a - l_{10} c + 10.$$

Again, $\quad \frac{b}{c} = \cos A$;

$$\therefore \quad l_{10} b = l_{10} c + L \cos A - 10.$$

$$\text{Or} \quad b^2 = c^2 - a^2 = (c + a)\ (c - a);$$

$$\therefore \quad l_{10} b = \tfrac{1}{2} \left\{ l_{10} (c + a) + l_{10} (c - a) \right\};$$

which determines b without previously finding A.

80. Different methods must be used in different cases to determine the unknown quantities; for what is said in Appendix III. 11. must always be carefully borne in mind, and such a formula selected as is calculated to give the result with the greatest practical degree of accuracy.

Thus in the last case, if b be very small, the angle A is nearly a right angle, and the increment of $\sin A$ corresponding to a small given increment δA of the angle,

$$\left[\text{i. e. } \cos A \,.\, \sin \delta A \,.\, \left\{1 - \tan A \,.\, \tan \frac{\delta A}{2}\right., \quad \text{Art. 60.}\right],$$

is inconsiderable, and does not, besides, vary nearly as the increment of the angle. In this case therefore the value of A cannot be determined from the Tables with any great degree of accuracy. (App. III. 11.)

The better way of determining A in such a case will be first to find the value of b, and then to determine A from its cosine.

$$\text{Thus} \quad \cos A = \frac{b}{c};$$

$$\begin{aligned} \therefore\ L \cos A &= l_{10} b - l_{10} c + 10, \\ &= l_{10} \sqrt{(c^2 - a^2)} - l_{10} c + 10, \\ &= \tfrac{1}{2} \,.\, \{l_{10}(c + a) + l_{10}(c - a)\} + 10 - l_{10} c. \end{aligned}$$

81. EXAMPLE. *Given* $c = 365{\cdot}1$, *and* $a = 348{\cdot}3$, *to find* A.

(The Logarithms employed in the Examples we shall give, will generally be found in the three pages of Logarithms, subjoined to Appendices I. and II.)

$$\text{Here,} \quad c + a = 713{\cdot}4,$$
$$c - a = 16{\cdot}8.$$

Performing the operations indicated in the last line of the last Article, we have

$$\begin{array}{rr} \mathrm{l}_{10}713{\cdot}4 = & 2{\cdot}8533331 \\ \mathrm{l}_{10}\ 16{\cdot}8 = & 1{\cdot}2253093 \\ \hline & 2)4{\cdot}0786424 \\ \hline & 2{\cdot}0393212 \\ & 10 \\ \hline & 12{\cdot}0393212 \\ \mathrm{l}_{10}365{\cdot}1 = & 2{\cdot}5624118 \\ \hline & 9{\cdot}4769094 \\ \text{And}\quad L\cos 72^0,\ 33' = & 9.4769380 \\ \hline \text{Difference} = & 286 \\ \hline \end{array}$$

See App. III. 9. Ex. 4.

Now diff. for 1″ is in this case 67·016;

$$\text{and}\quad \frac{286}{67{\cdot}016} = 4{\cdot}267 = 4{\cdot}27 \text{ nearly,}$$

$$\text{Therefore}\quad A = 72^0,\ 33',\ 4''{\cdot}27.$$

II. ON THE SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

82. *Let two angles and the side between them be given.* (A, C, b.)

Since $A + B + C = 180^0$,

$\therefore\ B = 180^0 - (A + C)$; which determines B.

Again, $a = b \cdot \dfrac{\sin A}{\sin B}$;

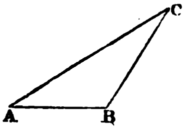
$\therefore\ \mathrm{l}_{10}a = \mathrm{l}_{10}b + L\sin A - L\sin B$; which determines a.

Again, $c = b \cdot \dfrac{\sin C}{\sin B}$;

$\therefore\ \mathrm{l}_{10}c = \mathrm{l}_{10}b + L\sin C - L\sin B$; which determines c.

If the value of B be not required; since $\sin B = \sin\{180^0 - (A + C)\} = \sin(A + C)$, we have

$$\mathrm{l}_{10}a = \mathrm{l}_{10}b + L\sin A - L\sin(A + C),$$



$$\text{and}\quad \mathrm{l}_{10}c = \mathrm{l}_{10}b + L\sin C - L\sin(A + C).$$

83. *Let two angles and a side opposite to one of them be given.* (A, C, a.)

Then, $B = 180^0 - (A + C)$.

Again, $b = a \cdot \dfrac{\sin B}{\sin A}$;

$$\therefore \; l_{10} b = l_{10} a + L \sin B - L \sin A.$$

$$\text{or} \quad = l_{10} a + L \sin (A + C) - L \sin A.$$

Again, $c = a \cdot \dfrac{\sin C}{\sin A}$;

$$\therefore \; l_{10} c = l_{10} a + L \sin C - L \sin A.$$

84. *Let two sides and the angle included between them be given.* (c, A, b.)

First, to determine B and C.

$$B + C = 180^0 - A; \qquad \therefore \; \tfrac{1}{2}(B + C) = \tfrac{1}{2} 180^0 - A).$$

Also, $\dfrac{b}{c} = \dfrac{\sin B}{\sin C}$; $\qquad \therefore \; \dfrac{\dfrac{b}{c} - 1}{\dfrac{b}{c} + 1} = \dfrac{\dfrac{\sin B}{\sin C} - 1}{\dfrac{\sin B}{\sin C} + 1}$;

$$\therefore \; \frac{b - c}{b + c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{\tan \frac{1}{2}(B - C)}{\tan \frac{1}{2}(B + C)}. \quad \text{Art. 52.}$$

And $\operatorname{Tan} \tfrac{1}{2}(B + C) = \tan \tfrac{1}{2}(180^0 - A) = \cot \dfrac{A}{2}$;

$$\therefore \; \operatorname{Tan} \tfrac{1}{2}(B - C) = \frac{b - c}{b + c} \cdot \cot \frac{A}{2};$$

$$\therefore \; L \tan \tfrac{1}{2}(B - C) = l_{10}(b - c) - l_{10}(b + c) + L \cot \frac{A}{2}. \quad (2).$$

$\frac{1}{2}(B - C)$ being thus determined, and $\frac{1}{2}(B + C)$ being known,

$$\left.\begin{array}{r} B = \tfrac{1}{2}(B + C) + \tfrac{1}{2}(B - C) \\ \text{and} \quad C = \tfrac{1}{2}(B + C) - \tfrac{1}{2}(B - C) \end{array}\right\} \text{ are known;}$$

And $a = c \cdot \dfrac{\sin A}{\sin C}$ determines a.

85. *The side* a *can be determined without previously determining* B *and* C. (c, A, b.)

$$\text{For}\quad a^2 = b^2 + c^2 - 2bc \,.\, \cos A$$

$$= b^2 + c^2 - 2bc \,.\left\{1 - 2\left(\sin\frac{A}{2}\right)^2\right\} \quad \text{Art. 39. (3).}$$

$$= (b-c)^2 + 4bc\,.\left(\sin\frac{A}{2}\right)^2$$

$$= (b-c)^2\,.\left\{1 + \frac{4bc}{(b-c)^2}\,.\left(\sin\frac{A}{2}\right)^2\right\}$$

$$= (b-c)^2\,.\left\{1 + \left(\frac{2\sqrt{bc}}{b-c}\,.\sin\frac{A}{2}\right)^2\right\}.$$

Now, $\frac{2\sqrt{bc}}{b-c}\,.\sin\frac{A}{2}$ may be of any magnitude and sign, and therefore there is *some* angle whose tangent is equal to this quantity. Let θ be this angle,

$$\text{Then }\ \operatorname{Tan}\theta = \frac{2\sqrt{bc}}{b-c}\,.\sin\frac{A}{2} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\text{and }\ a^2 = (b-c)^2\,.\left\{1 + (\tan\theta)^2\right\}$$

$$= (b-c)^2\,.(\sec\theta)^2$$

$$\therefore\ a = (b-c)\,.\sec\theta \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1), $L\tan\theta = \mathrm{l}_{10}2 + \tfrac{1}{2}\mathrm{l}_{10}b + \tfrac{1}{2}\mathrm{l}_{10}c - \mathrm{l}_{10}(b-c) + L\sin\frac{A}{2},$

$$= \tfrac{1}{2}(\mathrm{l}_{10}b + \mathrm{l}_{10}c) + \mathrm{l}_{10}2 + L\sin\frac{A}{2} - \mathrm{l}_{10}(b-c)\,;$$

which determines θ.

From (2), $\mathrm{l}_{10}a = \mathrm{l}_{10}(b-c) + L\sec\theta - 10$; which determines a.

86. If $b=c$ nearly, $b-c$ is very small, and $\frac{2\sqrt{bc}}{b-c}\,.\sin\frac{A}{2}$, the value of $\tan\theta$, is very large, (unless $\frac{A}{2}$, and therefore $\sin\frac{A}{2}$, be very small). And since the tangents of angles which are nearly right angles are very large, θ in this case is nearly a right angle.

Now if from its tangent we have to find an angle which does not consist of a certain number of degrees and minutes exactly, the additional seconds have to be determined on the principle that the increment of the tabular logarithm varies as the increment of the angle. But when the angle is equal to $(2n+1)\,.\,90^0$ nearly, this principle does not obtain for the tangent, Art. 63, Cor., App. III. 11; and therefore θ cannot in this case be determined with an accuracy sufficient to enable us, from the equation $a=(b-c)\,.\sec\theta$, to find a to any great degree of exactness.

When therefore c is nearly equal to b, and A is not a very small angle, the following method will give a with more exactness than the last Article does.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$= b^2 + c^2 - 2bc\,.\left\{2\left(\cos\frac{A}{2}\right)^2 - 1\right\} \quad \text{Art. 39. (2).}$$

$$= (b^2 + c^2 + 2bc) - 4bc\,.\left(\cos\frac{A}{2}\right)^2$$

$$= (b+c)^2\,.\left\{1 - \left(\frac{2\sqrt{bc}}{b+c}\,.\cos\frac{A}{2}\right)^2\right\}.$$

Now $(\sqrt{b}-\sqrt{c})^2$, or $b - 2\sqrt{bc} + c$, being a square, is necessarily a positive quantity; therefore the positive part of it is greater than the negative part, or $b+c > 2\sqrt{bc}$;

$\therefore \frac{2\sqrt{bc}}{b+c}$ is fractional, and *à fortiori* $\frac{2\sqrt{bc}}{b+c}\,.\cos\frac{A}{2}$ is fractional.

Let therefore ϕ be an angle such that

$$\text{Sin}\,\phi = \frac{2\sqrt{bc}}{b+2}\,.\cos\frac{A}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Then, $a^2 = (b+c)^2 . \{1 - (\sin\phi)^2\}$

$$\therefore\ a = (b+c)\,.\cos\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

From (1), $L\sin\phi = \text{l}_{10}2 + \frac{1}{2}\,\text{l}_{10}b + \frac{1}{2}\,\text{l}_{10}c - \text{l}_{10}(b+c) + L\cos\frac{A}{2}$,

$$= \tfrac{1}{2}\left(\text{l}_{10}b + \text{l}_{10}c\right) + \text{l}_{10}2 + L\cos\frac{A}{2} - \text{l}_{10}(b+c).$$

(2), $\text{l}_{10}a = \text{l}_{10}(b+c) + L\cos\phi - 10$;

which give ϕ and a respectively.

87. *Let two sides be given and an angle opposite to one of them.* (a, b, A.)

We have shewn in Art. 72, that with these data the solution is ambiguous, unless a be greater than b. But if a be greater than b, we then have

$\text{Sin}\,B = \dfrac{b}{a}\,.\sin A$, where B is an angle less than 90°.

Also, $C = 180^0 - (A+B)$; and, $c = a.\,\dfrac{\sin C}{\sin A}$.

88. *Let all the sides be given.* (a, b, c.)

Now, $\text{Sin}\,A = \dfrac{2}{bc}\sqrt{\{S.(S-a).(S-b).(S-c)\}}$,

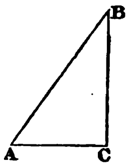
$$\text{Sin}\frac{A}{2} = \sqrt{\frac{(S-b).(S-c)}{bc}},$$

$$\text{Cos}\frac{A}{2} = \sqrt{\frac{S.(S-a)}{bc}},$$

$$\text{Tan}\frac{A}{2} = \sqrt{\frac{(S-b).(S-c)}{S.(S-a)}}.$$

If A be nearly 90^0, the first formula will not give the value of B very exactly, because the increment of $\sin A$ does not in that case vary as the increment of A, and it is also very small; App. III. 11;—in this case any one of the last three forms may be used. And the second or the third form must be taken, according as $\cos\frac{A}{2}$ or $\sin\frac{A}{2}$ is the greater, i. e. as $\frac{A}{2}$ is less or greater than 45^0. Art. 64. The fourth form is applicable in all cases except where $\frac{A}{2}$ is nearly 90^0. Art. 63.

89. Examples. 1. If BC be a perpendicular object standing on a horizontal plane, its height may be determined by measuring in that plane a line AC, which is called *a base*, and observing the angle BAC with a proper instrument.

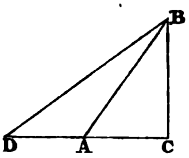


For $BC = AC \cdot \tan BAC$;

$$\therefore\ l_{10}BC = l_{10}AC + L\tan BAC - 10.$$

2. If it be not possible to come to the foot of the object, let a base AD be measured, such that the points D, A, C may be in the same line, and let the angles BDA, BAC be observed.

Here we have two angles given and a side of the $\triangle BAD$. By determining the side BA we can, from the right-angled triangle BAC, find the height of BC.

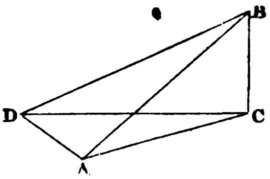


Thus, $$\frac{BA}{AD} = \frac{\sin BDA}{\sin DBA} = \frac{\sin BDA}{\sin(BAC - BDA)}.$$

And $$BC = BA \cdot \sin BAC = AD \cdot \frac{\sin BDA \cdot \sin BAC}{\sin(BAC - BDA)};$$

$$\therefore\ l_{10}BC = l_{10}AD + L\sin BDA + L\sin BAC - L\sin(BAC - BDA) - 10.$$

3. If D be not in the line AC, the height BC can yet be determined by taking proper observations.



At A let the angles BAC and BAD be observed, and at D the angle BDA.

Then in the $\triangle BDA$ we have the angles BDA, BAD and the side AD given. If then BA be determined from these data, we can find BC from the right-angled triangle BAC.

$$\text{Thus,}\quad \frac{BA}{AD} = \frac{\sin BDA}{\sin DBA} = \frac{\sin BDA}{\sin\{180^0 - (BDA + BAD)\}}$$

$$\therefore\ BA = AD \cdot \frac{\sin BDA}{\sin(BDA + BAD)}.$$

$$\text{And}\quad BC = BA \cdot \sin BAC = AD \cdot \frac{\sin BAC \cdot \sin BDA}{\sin(BDA + BAD)};$$

$$\text{l}_{10} BC = \text{l}_{10} AD + L\sin BAC + L\sin BDA - L\sin(BDA + BAD) - 10.$$

It is evident that this determination of CB is not affected by the circumstance of D lying out of the horizontal plane which passes through A and C. Hence it follows, that if a straight base AD be measured in *any* direction from A, and the angles BAC, BAD, BDA be observed, we shall be provided with sufficient data for finding the height of B above the horizontal plane passing through A.

4. Required to find the breadth of a river AD, from observations made from the top of a tower BC whose height is known. (Figure to Ex. 2.)

At B let the angles of depression of the points D and A below a horizontal line passing through B be observed. Since such a line will be parallel to CD, BDC and BAC are equal to these angles of depression.

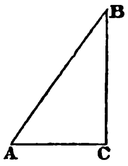
Hence we have,

$$DA = BA \cdot \frac{\sin DBA}{\sin BDA} = BA \cdot \frac{\sin (BAC - BDA)}{\sin BDA}$$

$$= \frac{BC}{\sin BAC} \cdot \frac{\sin (BAC - BDA)}{\sin BDA}.$$

By working a few examples of this kind the student will soon become familiar with the management of Trigonometrical formulæ. We add one more Example a little more difficult than the preceding, before leaving this part of our subject.

5. Required the error in height arising from a small given error in an observation of the angle in Example 1.



Let $BC = h$, $AC = a$, $\angle BAC = A$.

Let δh be the error in height produced by an error δA of the observed angle.

Then $h = a \cdot \tan A$,

$$h + \delta h = a \cdot \tan (A + \delta A);$$

$$\therefore \; \delta h = a \cdot \{\tan (A + \delta A) - \tan A\}$$

$$= a \cdot \left\{ \frac{\sin (A + \delta A)}{\cos (A + \delta A)} - \frac{\sin A}{\cos A} \right\}$$

$$= a \cdot \frac{\sin (A + \delta A) \cdot \cos A - \cos (A + \delta A) \cdot \sin A}{\cos (A + \delta A) \cdot \cos A}$$

$$= a \cdot \frac{\sin \{(A + \delta A) - A\}}{\cos (A + \delta A) \cdot \cos A}$$

$= a \cdot \dfrac{\sin \delta A}{(\cos A)^2}$. Since $\cos A = \cos (A + \delta A)$, nearly. Art. 61.

Cor. Hence we may determine when the error in height, arising from a small *given* error in the observed angle, will be the least.

$$\text{For}\quad \delta h = a\,.\,\frac{\sin\delta A}{(\cos A)^2} = \frac{h}{\tan A}\,.\,\frac{\sin\delta A}{(\cos A)^2}$$

$$= \frac{h\,.\sin\delta A}{\sin A\,.\cos A} = \frac{2h\,.\sin\delta A}{2\sin A\,.\cos A}$$

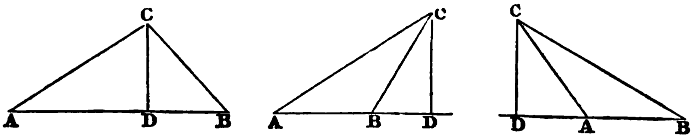
$$= \frac{2h\,.\sin\delta A}{\sin 2A}.$$

Now h being constant, and δA being given, this expression, which is the error in altitude, will be the least when $\sin 2A$ is the greatest,—that is, when $2A = 90^0$, or $A = 45^0$.

The observer therefore ought to move along AC till $\angle BAC = 45^0$, and then by measuring AC he will determine CB, which is in this case equal to AC, with the least chance of error.

90. *To find the area of a triangle, the sides being given.*

Let ABC be the triangle, and from C draw CD perpendicular to AB, or to AB produced either way.



Then Area of triangle ABC, being half the rectangle on the same base and between the same parallels,

$$= \tfrac{1}{2} AB\,.\,CD = \tfrac{1}{2} AB\,.\,AC\,.\sin CAB$$

$$= \tfrac{1}{2} cb\,.\,\frac{2}{bc}\,.\sqrt{\{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)\}}.\quad \text{Art. 76. Cor. 1.}$$

$$= \sqrt{\{S\,.\,(S-a)\,.\,(S-b)\,.\,(S-c)\}}.$$

91. *The area of the triangle also* $= \frac{1}{2} a^2 . \frac{\sin B . \sin C}{\sin (B + C)}$.

For $\text{Area} = \frac{1}{2} AB . CD$

$$= \frac{1}{2} . AB . BC . \sin B$$

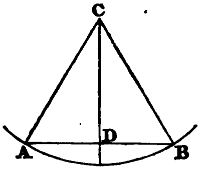
$$= \frac{1}{2} BC . \frac{\sin C}{\sin A} . BC . \sin B$$

$$= \frac{1}{2} a^2 . \frac{\sin B . \sin C}{\sin (B + C)};$$

since $\operatorname{Sin} A = \sin \{180^0 - (B + C)\} = \sin (B + C)$.

92. ***To find the radii of the circles described within and about a regular polygon of any number of sides.***

Let AB be a side of a regular polygon of n sides.



Since the polygon is regular, it may have a circle inscribed in and about it, and each of the sides will subtend the same angle at the common centre C of these circles.

Draw CD perpendicular to AB.

Then $AD = DB$, and CD is the radius of the inscribed circle.

$$\text{Let} \quad r = CD, \text{ and } R = AC.$$

Now the sum of all the angles which the sides subtend at C

$$= n \times \angle ACB = 360^0;$$

$$\therefore \angle ACB = \frac{360^0}{n}; \qquad \therefore \angle ACD = \frac{\angle ACB}{2} = \frac{180^0}{n};$$

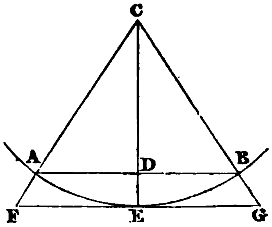
$$\text{And } \frac{AD}{CD} = \tan ACD;$$

$$\therefore r, = CD, = AD . \cot ACD = \frac{1}{2} AB . \cot \frac{180^0}{n} \ldots\ldots\ldots\ldots (1).$$

Again, $R = AC = \dfrac{AD}{\sin ACD} = \dfrac{\frac{1}{2}AB}{\sin \dfrac{180^\circ}{n}} = \frac{1}{2}AB \,.\, \operatorname{cosec}\dfrac{180^\circ}{n} \ldots (2)$.

93. *To find the area of a regular polygon of any number of sides which is inscribed within or about a circle of given radius.*

Let AB be an arc of the circle whose centre is C; AB a side of the inscribed regular polygon of n sides; CE at right angles to AB, and therefore bisecting it; FG a tangent through E; then FG is a side of the circumscribed polygon of n sides. Let $CA = r$.



$$ADB = 2AD = 2 \,.\, r \,.\, \sin ACD = 2r \,.\, \sin \frac{360^\circ}{2n} = 2r \,.\, \sin \frac{180^\circ}{n},$$

$$FG = 2FE = 2r \,.\, \tan \frac{180^\circ}{n}.$$

A = area of inscribed polygon

$$= n \,.\, \Delta CAB$$

$$= n \,.\, \frac{CA \,.\, CB \,.\, \sin ACB}{2}$$

$$= \frac{n}{2} r^2 \,.\, \sin \frac{360^\circ}{n}.$$

A' = area of circumscribed polygon

$$= n \,.\, \Delta CFG$$

$$= n \,.\, CE \,.\, FE$$

$$= n \,.\, r^2 \,.\, \tan \frac{180^\circ}{n}.$$

Cor. These areas may be thus compared;

$$\frac{\text{Area of inscribed polygon}}{\text{Area of circumscribed polygon}} = \frac{\Delta CAD}{\Delta CFE}$$

$$= \frac{CD^2}{CE^2}, \text{ since they are similar,}$$

$$= \left(\frac{CD}{CA}\right)^2$$

$$= \left(\cos\frac{180^0}{n}\right)^2.$$

94. *To find the area of a regular polygon of* n *sides in terms of a side of the polygon.* (Fig. Art. 92.)

AB being a side of the polygon, we have,

$$\text{Area} = n\,.\,\Delta CAB$$

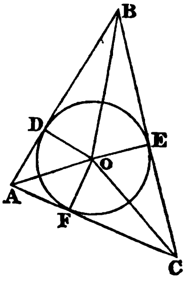
$$= n\,.\,\frac{AB\,.\,DC}{2}$$

$$= \frac{n}{2}\,.\,AB\,.\,AD\,.\cot ACD$$

$$= \frac{n}{2}\,.\,AB\,.\,\frac{AB}{2}\,.\cot\frac{180^0}{n}$$

$$= \frac{n}{4}\,.\,(AB)^2.\cot\frac{180^0}{n}.$$

95. *To find the radii of the circles described in and about a triangle whose sides are given.*



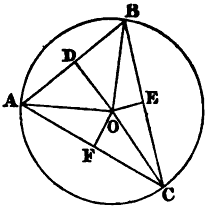
Let the lines bisecting the angles A and B meet in O, and from O draw OD, OE, OF perpendiculars to the sides.

Then, Euclid IV. 4, O is the centre of the inscribed circle, and r its radius $= OD = OE = OF$.

Now, Area of $\triangle ABC = \triangle AOB + \triangle BOC + \triangle COA$;

$$\therefore \sqrt{\{S.(S-a).(S-b).(S-c)\}} = r.\frac{c}{2} + r.\frac{a}{2} + r.\frac{b}{2} = r.S;$$

$$\therefore r = \sqrt{\frac{(S-a).(S-b).(S-c)}{S}} \quad\ldots\ldots\ldots\ldots\ldots (1).$$



Again; bisect the sides of the triangle in D, E, F, and draw perpendiculars which will meet in a point O which is the centre of the circumscribed circle; and R, its radius, $= OA = OB = OC$. Euclid IV. 5.

Area of $ABC = \frac{1}{2} AC.AB.\sin A$, Art. 90.

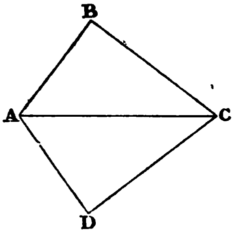
And $A = \frac{1}{2} BOC$, Euclid III. 20;

$$\therefore \text{Sin } A = \sin\tfrac{1}{2} BOC = \sin BOE = \frac{BE}{BO} = \frac{\frac{1}{2}a}{R};$$

$$\therefore \frac{2}{bc}\sqrt{\{S.(S-a).(S-b).(S-c)\}} = \frac{\frac{1}{2}a}{R};$$

$$\therefore R = \frac{abc}{4\sqrt{\{S.(S-a).(S-b).(S-c)\}}} \quad\ldots\ldots\ldots (2).$$

96. *To find the area of a quadrilateral figure whose opposite angles are supplements to each other.*



Let $ABCD$ be the quadrilateral.

Let $AB = a$, $BC = b$, $CD = c$, $DA = d$;

Join AC.

$$\begin{aligned}\text{Area } ABCD &= \triangle ABC + \triangle ADC \\ &= \tfrac{1}{2}ab.\sin B + \tfrac{1}{2}cd.\sin D \\ &= \tfrac{1}{2}(ab + cd).\sin B.\end{aligned}$$

since $\text{Sin } D = \sin(180^0 - D) = \sin B$.

Now from ΔABC,

$$a^2 + b^2 - AC^2 = 2ab \cdot \cos B.$$

And from ΔABC,

$$c^2 + d^2 - AC^2 = 2cd \cdot \cos D = -2cd \cdot \cos B;$$

$\therefore$ by subtracting,

$$a^2 + b^2 - c^2 - d^2 = 2(ab + cd) \cdot \cos B;$$

$$\therefore (\sin B)^2 = 1 - (\cos B)^2 = 1 - \left\{\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}\right\}^2$$

$$= \frac{4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}.$$

And $(\text{area } ABCD)^2 = \frac{1}{4}(ab + cd)^2 \cdot (\sin B)^2$;

$$\therefore = \frac{1}{16} \cdot \{4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2\}$$

$$= \frac{1}{16} \cdot \{2(ab+cd) + (a^2+b^2-c^2-d^2)\} \cdot \{2(ab+cd) - (a^2+b^2-c^2-d^2)\}$$

$$= \frac{1}{16} \cdot \{(a + b)^2 - (c - d)^2\} \cdot \{c + d)^2 - (a - b)^2\}$$

$$= \frac{1}{16} \cdot (a+b+c-d) \cdot (a+b+d-c) \cdot (c+d+a-b) \cdot (c+d+b-a).$$

If $S = \frac{1}{2}(a + b + c + d)$, this equation becomes

$$\text{Area } ABCD = \sqrt{\{(S - a) \cdot (S - b) \cdot (S - c) \cdot (S - d)\}}.$$

CHAPTER V.

ON THE USE OF SUBSIDIARY ANGLES.

97. **Def.** An angle which is introduced to assist Trigonometrical calculation, (as θ and ϕ in Arts. 85, 86,) is called a *Subsidiary angle.*

98. Subsidiary angles are of great use and extensive application, particularly in adapting the formulæ of Astronomy to logarithmic calculation.

Examples. (1). *Having given* a, a, *and* δ, *to determine* l *in a form adapted to logarithmic calculation from the equation,*

$$\text{Sin } a = \cos l \, . \cos \delta \, . \cos a + \sin l \, . \sin \delta \, ;$$

(*Maddy's Astronomy by Hymers*, Art. 118.)

The equation may be written,

$$\text{Sin } a = \sin \delta \, . \left(\sin l + \cos l \, . \frac{\cos \delta \, . \cos a}{\sin \delta}\right) .$$

Now since tangents are of any magnitude and sign, there is *some* angle ϕ, such that

$$\text{Tan } \phi, \quad \text{or } \frac{\sin \phi}{\cos \phi}, = \frac{\cos \delta \, . \cos a}{\sin \delta} = \cot \delta \, . \cos a \ldots\ldots\ldots (1);$$

$$\therefore \; \text{Sin } a = \sin \delta \, . \left(\sin l + \cos l \, . \frac{\sin \phi}{\cos \phi}\right)$$

$$= \frac{\sin \delta}{\cos \phi} \, . \, (\sin l \, . \cos \phi + \cos l \, . \sin \phi)$$

$$= \frac{\sin \delta}{\cos \phi} \, . \sin (l + \phi),$$

$$\therefore \; \text{Sin } (l + \phi) = \frac{\sin a \, . \cos \phi}{\sin \delta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1), $L\tan\phi = L\cot\delta + L\cos a - 10$, which gives ϕ.

(2), $L\sin(l+\phi) = L\sin a + L\cos\phi - L\sin\delta$,

which gives $l+\phi$, and thence l.

(2). *To express* $a.\cos\theta + b.\cos(\theta + a)$ *under the form* $A.\cos(B+\theta)$.

$$\text{Let}\quad \rho = a.\cos\theta + b.\cos(\theta + a)$$
$$= a.\cos\theta + b.\cos a.\cos\theta - b.\sin a.\sin\theta,$$
$$= a.(1 + \frac{b}{a}.\cos a).\cos\theta - b.\sin a.\sin\theta.$$

Let ϕ be the angle whose tangent is $\frac{b}{a}.\cos a$,

$$\text{or,}\quad \text{Tan}\,\phi = \frac{b}{a}.\cos a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\therefore\ 1 + \frac{b}{a}.\cos a = 1 + \tan\phi = \frac{\cos\phi + \sin\phi}{\cos\phi}$$
$$= \sqrt{2}.\frac{\frac{1}{\sqrt{2}}.\cos\phi + \frac{1}{\sqrt{2}}.\sin\phi}{\cos\phi}$$
$$= \sqrt{2}.\frac{\sin 45^0.\cos\phi + \cos 45^0.\sin\phi}{\cos\phi}$$
$$= \sqrt{2}.\frac{\sin(45^0+\phi)}{\cos\phi};$$

$$\therefore\ \rho = a.\sqrt{2}.\frac{\sin(45^0+\phi)}{\cos\phi}.\left\{\cos\theta - \frac{b.\sin a.\cos\phi}{a.\sqrt{2}.\sin(45^0+\phi)}.\sin\theta\right\}.$$

$$\text{Let}\quad \text{Tan}\,\phi' = \frac{b.\sin a.\cos\phi}{a.\sqrt{2}.\sin(45^0+\phi)}, \ldots\ldots\ldots\ldots(2).$$

$$\text{Then}\quad \rho = a\sqrt{2}.\frac{\sin(45^0+\phi)}{\cos\phi.\cos\phi'}.\cos(\phi'+\theta);\ldots\ldots\ldots\ldots(3).$$

A result of the required form.

(3). *Adapt* $\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}$ *to logarithmic computation, by means of a subsidiary angle.*

$$\sqrt{\frac{a-b}{a+b}}+\sqrt{\frac{a+b}{a-b}}=\frac{a-b+a+b}{\sqrt{(a^2-b^2)}}$$

$$=\frac{2a}{\sqrt{(a^2-b^2)}}=\frac{2}{\sqrt{\left(1-\frac{b^2}{a^2}\right)}}.$$

Now $\frac{b^2}{a^2}$ is necessarily a positive quantity, and it may be of any magnitude, let therefore θ be an angle such that

$$(\mathrm{Tan}\,\theta)^2=\frac{b^2}{a^2}; \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\therefore \quad \frac{2}{\sqrt{\left(1-\frac{b^2}{a^2}\right)}}=\frac{2}{\sqrt{1-\left(\frac{\sin\theta}{\cos\theta}\right)^2}}=\frac{2\cos\theta}{\sqrt{(\cos\theta)^2-(\sin\theta)^2}}$$

$$=\frac{2\cos\theta}{(\cos 2\theta)^{\frac{1}{2}}};\text{ a result of the required form}\ldots\ldots(2).$$

(4). *To adapt* $\sqrt{\{(a^2+c^2)^2-(a^2-c^2)^2+(a^2+c^2-b^2)^2\}}$ *to logarithmic computation.*

Let the expression $=\rho$, then

$$\rho=a^2.\left(1+\frac{c^2}{a^2}\right).\sqrt{\left\{1-\frac{\left(1-\frac{c^2}{a^2}\right)^2}{\left(1+\frac{c^2}{a^2}\right)^2}-\left(1-\frac{b^2}{a^2.\left(1+\frac{c^2}{a^2}\right)}\right)^2\right\}}.$$

$$\text{Let}\quad \frac{c}{a}=\tan\theta\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\therefore\;\rho=a^2.(\sec\theta)^2.\sqrt{\left\{1-\left((\cos\theta)^2-(\sin\theta)^2\right)^2-\left(1-\frac{b^2}{a^2(\sec\theta)^2}\right)^2\right\}}.$$

Let $$\frac{b}{a \cdot \sec\theta} = \tan\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\therefore\ \rho = a^2 . (\sec\theta)^2 . \surd\left[1 - (\cos 2\theta)^2 - \{1 - (\tan\phi)^2\}^2\right]$$

$$= a^2 . (\sec\theta)^2 . \surd\left\{(\sin 2\theta)^2 - \frac{(\cos 2\phi)^2}{(\cos\phi)^4}\right\},$$

$$= a^2 . (\sec\theta)^2 . \sin 2\theta . \surd\left\{1 - \frac{(\cos 2\phi)^2}{(\sin 2\theta)^2 . (\cos\phi)^4}\right\};$$

Let $$\frac{\cos 2\phi}{\sin 2\theta . (\cos\phi)^2} = \tan\phi' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

And since $(\sec\theta)^2 . \sin 2\theta = \dfrac{1}{(\cos\theta)^2} . 2\sin\theta . \cos\theta = 2\tan\theta$;

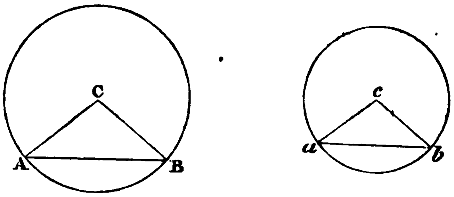
$$\therefore\ \rho = 2\,a^2 . \tan\theta . \frac{(\cos 2\phi')^{\frac{1}{2}}}{\cos\phi'} \;\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

CHAPTER VI.

ANALYTICAL TRIGONOMETRY.

99. *The circumference of a circle varies as its radius.*

Let P, p be the perimeters of two regular polygons of n sides each, which are inscribed in two circles whose radii are R, r and circumferences C, c respectively.



$$\text{Let}\quad X = C - P,\quad x = c - p.$$

Now as n increases, P and p increase; therefore X and x are variable quantities dependent on the value of n.

Let AB, ab be sides of the polygons; C, c the centres of the circles.

$$\text{Then}\quad \angle ACB = \frac{360^0}{n} = \angle acb,$$

and ACB, acb are similar triangles;

$$\therefore\ \frac{R}{r} = \frac{AB}{ab} = \frac{n\,.\,AB}{n\,.\,ab} = \frac{P}{p} = \frac{C-X}{c-x},$$

$$\therefore\ R\,.\,c - R\,.\,x = r\,.\,C - r\,.\,X.$$

Now since $R\,.\,c$, $r\,.\,C$ are constant, and $R\,.\,x$, $r\,.\,X$ are variable, we must have, *Francœur's Pure Mathematics*, Art. 167,

$$R\,.\,c = r\,.\,C;$$

$$\therefore\ \frac{c}{C} = \frac{r}{R},\quad \text{or } c \propto r.$$

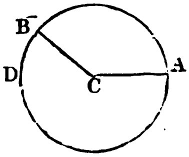
COR. 1. Since $c \propto r$; $\therefore \frac{c}{r}$ is a constant quantity.

This constant quantity is always represented by 2π; the approximate value of π, (3·14159...,) will be determined hereafter.

COR. 2. Since $\frac{c}{r} = 2\pi$; $\therefore c = 2\pi r$; or $2\pi r$ represents the circumference of a circle whose radius is r.

100. *If an arc be traced out by a point in the line* CB, *by whose revolution from the position* CA *an angle* ACB *is described, the angle* ACB *may be properly measured by the ratio* $\frac{\text{arc AB}}{\text{radius AC}}$.

Since in equal circles, and therefore in the same circle, angles at the centre have the same ratio to each other as the arcs on which they stand, Euclid VI. 33;



$$\therefore \frac{\angle ACB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{whole circumference } ABDA}$$

$$= \frac{AB}{2\pi . CA}. \quad \text{Art. 99. Cor. 2;}$$

$$\therefore \angle ACB = \frac{4 \text{ right angles}}{2\pi} . \frac{AB}{CA}.$$

Now, since $\frac{4 \text{ right angles}}{2\pi}$ is a constant quantity, $\frac{AB}{CA}$ increases or decreases in proportion as the angle ACB increases or decreases; therefore $\frac{\text{arc}}{\text{radius}}$ is a proper measure of the magnitude of an angle, and we may say that the angle is ***equal*** to $\frac{\text{arc}}{\text{radius}}$.

COR. Hence, if $\theta = \frac{\text{arc}}{\text{radius}}$, then Arc $= r\theta$.

101. In the preceding Chapters we have measured the magnitude of an angle by the number of times it contains a fixed and definite angle called a degree, and its subdivisions;—for several *analytical* investigations, however, the circular measure $\frac{\text{arc}}{\text{radius}}$ is much more convenient.

When this circular measure is used, we shall, generally speaking, denote angles by the letters of the Greek Alphabet.

102. *Having given the circular measure* $\frac{\text{arc}}{\text{radius}}$ *of an angle, to determine how many degrees the angle contains; and conversely.*

Let θ be the circular measure of the angle which contains A degrees.

Since $\frac{\text{circumference}}{\text{radius}} = 2\pi$, and the circumference subtends four right angles, therefore 2π is the circular measure of four right angles.

$$\text{Now} \quad \frac{A^0}{360^0} = \frac{\text{given angle}}{4 \text{ right angles}} = \frac{\theta}{2\pi};$$

$$\therefore \quad \theta = 2\pi \cdot \frac{A}{360} = \frac{\pi}{180} \cdot A \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$= \cdot 017453 \ldots \times A.$$

$$\text{And} \quad A = 360 \cdot \frac{\theta}{2\pi} = \frac{180}{\pi} \cdot \theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$= 57 \cdot 29577 \ldots \times \theta.$$

Ex. 1. If $A = 60$,

$$\theta = \frac{60}{180} \cdot \pi = \tfrac{1}{3} \times 3 \cdot 14159 \ldots = 1 \cdot 04719 \ldots\ldots$$

Ex. 2. *Required the number of degrees subtended by the arc which is equal to the radius.*

$$\text{In this case } \theta = \frac{\text{arc}}{\text{radius}} = 1;$$

$$\therefore\ A = \frac{180}{\pi}.\theta = \frac{180^0}{3{\cdot}14159..} = 57^0{\cdot}29577$$

$$\begin{array}{r} 60 \\ \hline 17{\cdot}7462 \\ 60 \\ \hline 44{\cdot}772 \\ \hline \end{array}$$

And the degrees, minutes, and seconds required are

$$57^0,\ 17',\ 44''{\cdot}77.$$

Cor. It is often necessary to know the number of *seconds* subtended by an arc which is equal to the radius.

$$\begin{array}{rr} \text{The number of degrees subtended} = & 57{\cdot}29577 \\ & 60 \\ \hline \text{minutes} & 3437{\cdot}7462 \\ & 60 \\ \hline \end{array}$$

number of seconds subtended $= 206264{\cdot}772 = 206264{\cdot}8$ nearly.

103. *Comparison of the unit of measurement in the two methods of estimating the magnitude of an angle.*

If the arc subtending the angle be equal in length to the radius with which it is described, the expression $\frac{\text{arc}}{\text{radius}}$ becomes 1, and therefore *this* is the angle which is the unit of measurement when the magnitudes of angles are estimated by the circular measure.

Now this angle, as it appears from Ex. 2, of the last Article, contains the angle which is called a degree 57·29577 times; and therefore the one of these angular units is 57·29577 times greater than the other.

104. Four right angles being represented by 2π, and therefore two right angles by π, we have from Arts. 24, 25, if the angle A^0 be represented according to the circular measure by θ,

$$\begin{cases} \sin\theta = \sin(2n\pi+\theta), & \text{or } -\sin(2n\pi-\theta). \\ \sin\theta = -\sin\{(2n+1)\pi+\theta\}, & \text{or } \sin\{(2n+1)\pi-\theta\}. \end{cases}$$

$$\begin{cases} \cos\theta = \cos(2n\pi+\theta), & \text{or } \cos(2n\pi-\theta). \\ \cos\theta = -\cos\{(2n+1)\pi+\theta\}, & \text{or } -\cos\{(2n+1)\pi-\theta\}. \end{cases}$$

$$\begin{cases} \tan\theta = \tan(2n\pi+\theta), & \text{or } -\tan(2n\pi-\theta). \\ \tan\theta = \tan\{(2n+1)\pi+\theta\}, & \text{or } -\tan\{(2n+1)\pi-\theta\}. \end{cases}$$

$$\begin{cases} \sec\theta = \sec\{2n\pi+\theta\}, & \text{or } \sec(2n\pi-\theta). \\ \sec\theta = -\sec\{(2n+1)\pi+\theta\}, & \text{or } -\sec\{(2n+1)\pi-\theta\}. \end{cases}$$

105. *If* $x+\frac{1}{x}$ *represent* $2\cos\theta$, *then* $x-\frac{1}{x}$ *will represent* $2\sqrt{-1}\sin\theta$.

For $$-(\sin\theta)^2 = (\cos\theta)^2 - 1 = \tfrac{1}{4}.\left(x+\frac{1}{x}\right)^2 - 1;$$

$$\therefore\ -4(\sin\theta)^2 = x^2 - 2 + \frac{1}{x^2};$$

$$\therefore\ 2\sqrt{-1}\sin\theta = x - \frac{1}{x}.$$

106. Demoivre's Theorem. *To shew that for any value of* m,

$$(\cos\theta \pm \sqrt{-1}\,\sin\theta)^m = \cos m\theta \pm \sqrt{-1}\,\sin m\theta.$$

$$(\text{Cos}\,\theta \pm \sqrt{-1}\,\sin\theta).(\cos\theta \pm \sqrt{-1}\,\sin\theta)$$

$$= (\cos\theta)^2 - (\sin\theta)^2 \pm \sqrt{-1}.2\sin\theta.\cos\theta.$$

Or, $$(\cos\theta \pm \sqrt{-1}\,\sin\theta)^2 = \cos 2\theta \pm \sqrt{-1}\,\sin 2\theta.$$

Again, $(\cos\theta \pm \sqrt{-1}\ \sin\theta)^2 . (\cos\theta \pm \sqrt{-1}\ \sin\theta)$

$$= (\cos 2\theta \pm \sqrt{-1}\ \sin 2\theta) . (\cos\theta \pm \sqrt{-1}\ \sin\theta)$$

$$= (\cos 2\theta . \cos\theta - \sin 2\theta . \sin\theta) \pm \sqrt{-1} . (\sin 2\theta . \cos\theta + \cos 2\theta . \sin\theta);$$

$$\therefore\ (\cos\theta \pm \sqrt{-1}\ \sin\theta)^3 = \cos 3\theta \pm \sqrt{-1}\ \sin 3\theta.$$

Suppose this law to hold for m factors, so that

$$(\cos\theta \pm \sqrt{-1}\ \sin\theta)^m = \cos m\theta \pm \sqrt{-1}\ \sin m\theta;$$

$$\therefore\ (\cos\theta \pm \sqrt{-1}\ \sin\theta)^{m+1}$$

$$= (\cos m\theta \pm \sqrt{-1}\ \sin m\theta) . (\cos\theta \pm \sqrt{-1}\ \sin\theta)$$

$$= \cos m\theta . \cos\theta - \sin m\theta . \sin\theta \pm \sqrt{-1} . (\sin m\theta . \cos\theta + \cos m\theta . \sin\theta)$$

$$= \cos (m+1)\,\theta \pm \sqrt{-1} . \sin (m+1)\,\theta.$$

If therefore the law hold for m factors, it holds for $m+1$ factors; but we have shewn it to hold when $m = 3$, it therefore holds when $m = 4$, and by successive inductions we conclude it to be true when the index is any positive integer.

Again, $(\cos\theta \pm \sqrt{-1}\ \sin\theta)^{-m} = \left\{\dfrac{1}{\cos\theta \pm \sqrt{-1}\ \sin\theta}\right\}^m$

$$= \left\{\frac{(\cos\theta)^2 + (\sin\theta)^2}{\cos\theta \pm \sqrt{-1}\ \sin\theta}\right\}^m,$$

$= (\cos\theta \mp \sqrt{-1}\ \sin\theta)^m$, by actual division;

$$= \cos m\theta \mp \sqrt{-1}\ \sin m\theta,$$

$$= \cos(-m\theta) \pm \sqrt{-1}\ \sin(-m\theta);$$

which proves the theorem for negative indices.

Again, $(\cos\theta \pm \sqrt{-1}\sin\theta)^m = \cos m\theta \pm \sqrt{-1}\sin m\theta.$

But $(\cos\frac{m}{n}\theta \pm \sqrt{-1}\sin\frac{m}{n}\theta)^n = \cos m\theta \pm \sqrt{-1}\sin m\theta;$

$$\therefore\ (\cos\theta \pm \sqrt{-1}\sin\theta)^m = (\cos\frac{m}{n}\theta \pm \sqrt{-1}\sin\frac{m}{n}\theta)^n;$$

$$\therefore\ (\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{m}{n}} = \cos\frac{m}{n}\theta \pm \sqrt{-1}\sin\frac{m}{n}\theta;$$

which proves the theorem for fractional indices.

Cor. By the theorem just proved,

$$(\cos\phi \pm \sqrt{-1}\sin\phi)^m = \cos m\phi \pm \sqrt{-1}\sin m\phi,$$

m being positive or negative, whole or fractional.

Let $\phi = 2p\pi + \theta$, where p is any integer;

$$\therefore\ \left.\begin{aligned}\cos\phi &= \cos(2p\pi + \theta) = \cos\theta,\\ \sin\phi &= \sin(2p\pi + \theta) = \sin\theta.\end{aligned}\right\}\quad \text{Art. 104.}$$

1. Let the index of $(\cos\theta \pm \sqrt{-1}\sin\theta)$ be integral, as m;

$$\text{Then}\quad \begin{aligned}\cos m\phi &= \cos(2mp\pi + m\theta) = \cos m\theta,\\ \sin m\phi &= \sin(2mp\pi + m\theta) = \sin m\theta.\end{aligned}$$

2. Let the index be fractional, as $\frac{m}{n}$;

Then p, being an integer, may be represented by $qn + r$, where q may be 0 or any integer, and r may be 0 or any integer less than n;

$$\begin{aligned}\therefore\ \cos\frac{m}{n}\phi = \cos\frac{m}{n}(2p\pi + \theta) &= \cos\frac{m}{n}\{2(qn + r)\pi + \theta\}\\ &= \cos\{mq\,.\,2\pi + \frac{m}{n}(2r\pi + \theta)\}\\ &= \cos\frac{m}{n}(2r\pi + \theta).\end{aligned}$$

Similarly, $\sin\frac{m}{n}\phi = \sin\frac{m}{n}(2r\pi + \theta).$

From these two cases therefore it appears that the theorem might have been thus enunciated;

If the index be an integer,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^m = \cos m\theta \pm \sqrt{-1}\sin m\theta;$$

If the index be fractional,

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{m}{n}} = \cos\frac{m}{n}(2r\pi+\theta) \pm \sqrt{-1}\sin\frac{m}{n}(2r\pi+\theta);$$

where r *is* 0, *or any integer less than* n.

Note. It is to be observed that by giving r all values from 0 to $n-1$, we obtain from the second member of the second equation, the n different values of

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^{\frac{m}{n}}.$$

107. *If* $2\cos\theta$ *be represented by* $x+\frac{1}{x}$, *then* $2\cos m\theta$ *will be represented by* $x^m+\frac{1}{x^m}$, *and* $2\sqrt{-1}\sin m\theta$ *by* $x^m-\frac{1}{x^m}$.

$$\text{For } 2\cos\theta = x+\frac{1}{x},$$

$$\text{And } \therefore\ 2\sqrt{-1}\sin\theta = x-\frac{1}{x}, \qquad \text{Art. 105.}$$

$\therefore$ by addition and subtraction,

$$\begin{cases} x = \cos\theta + \sqrt{-1}\sin\theta, \\ \dfrac{1}{x} = \cos\theta - \sqrt{-1}\sin\theta, \end{cases}$$

$$\therefore \begin{cases} x^m = (\cos\theta + \sqrt{-1}\sin\theta)^m = \cos m\theta + \sqrt{-1}\sin m\theta, \\ \dfrac{1}{x^m} = (\cos\theta - \sqrt{-1}\sin\theta)^m = \cos m\theta - \sqrt{-1}\sin m\theta, \end{cases}$$

$$\therefore \begin{cases} x^m + \dfrac{1}{x^m} = 2\cos m\theta, \\ x^m - \dfrac{1}{x^m} = 2\sqrt{-1}\sin m\theta. \end{cases}$$

Cor. By making use of the equations of the corollary to Art. 106, we have,

(1). The index being an integer,

$$2\cos m\theta = x^m + \frac{1}{x^m}; \qquad 2\sqrt{-1}\sin m\theta = x^m - \frac{1}{x^m};$$

(2). The index being a fraction,

$$2\cos\frac{m}{n}(2r\pi + \theta) = x^{\frac{m}{n}} + \frac{1}{x^{\frac{m}{n}}}; \quad 2\sqrt{-1}\sin\frac{m}{n}(2r\pi + \theta) = x^{\frac{m}{n}} - \frac{1}{x^{\frac{m}{n}}};$$

where r is 0 or any integer less than n.

108. *To express any positive integral power of the cosine of an angle in terms of the cosines of the multiples of the angle.*

Let $2\cos\theta = x + \dfrac{1}{x}$; $\therefore$ $2\cos n\theta = x^n + \dfrac{1}{x^n}$. Art. 107.

$$\text{Now } (2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$$

$$= x^n + n \,.\, x^{n-2} + n \,.\, \frac{n-1}{2} x^{n-4} + \dots$$

$$+ n \,.\, \frac{n-1}{2} \,.\, \frac{1}{x^{n-4}} + n \,.\, \frac{1}{x^{n-2}} + \frac{1}{x^n} \qquad (1).$$

$$= \left(x^n + \frac{1}{x^n}\right) + n \,.\, \left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \dots\dots \qquad (2).$$

1. If n be even; the last term of (2), or the $(\frac{1}{2}n+1)^{\text{th}}$ term of (1), is

$$n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-\frac{1}{2}n+1}{\frac{1}{2}n}.$$

2. If n be odd; the sum of the two middle terms of (1), viz.

the $\{\frac{1}{2}(n-1)+1\}^{\text{th}}$ and the $\{\frac{1}{2}(n-1)+2\}^{\text{th}}$,

(which have the same coefficients), is

$$n \cdot \frac{n-1}{2} \cdots \frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)} \cdot \left(x+\frac{1}{x}\right).$$

Hence (2) becomes

$$(2\cos\theta)^n = 2\cos n\theta + n \cdot 2\cos(n-2)\theta + \ldots$$

$$+\begin{cases} n \cdot \dfrac{n-1}{2} \cdots \dfrac{n-\frac{1}{2}n+1}{\frac{1}{2}n}; & n \text{ even.} \\ n \cdot \dfrac{n-1}{2} \cdots \dfrac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)} \cdot 2\cos\theta; & n \text{ odd.} \end{cases}$$

$$\therefore\ 2^{n-1} \cdot (\cos\theta)^n = \cos n\theta + n \cdot \cos(n-2)\theta$$

$$+ n \cdot \frac{n-1}{2} \cdot \cos(n-4)\theta + \ldots$$

the last term being

$$+\tfrac{1}{2} \cdot n \cdot \frac{n-1}{2} \cdots \frac{n-\frac{1}{2}n+1}{\frac{1}{2}n}; \quad n \text{ even.}$$

$$+ n \cdot \frac{n-1}{2} \cdots \frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)} \cdot \cos\theta; \quad n \text{ odd.}$$

Thus, if $n=2$, we have $2(\cos\theta)^2 = \cos 2\theta + 1$,

$n=3$, ... $4(\cos\theta)^3 = \cos 3\theta + 3\cos\theta$,

$n=4$, ... $8(\cos\theta)^4 = \cos 4\theta + 4\cos 2\theta + 3$.

109. *To express any positive integral power of the sine of an angle in terms of the sines and cosines of the multiples of the angle.*

Let $2\sqrt{-1}\sin\theta = x - \frac{1}{x}$; $\quad\therefore\ 2\sqrt{-1}\sin n\theta = x^n - \frac{1}{x^n}$.

$$\text{And}\ (2\sqrt{-1}\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$$

$$= x^n - n.x^{n-2} + n.\frac{n-1}{2}.x^{n-4} - \ldots \pm n.\frac{n-1}{2}.\frac{1}{x^{n-4}} \mp n.\frac{1}{x^{n-2}} \pm \frac{1}{x^n} \ldots (1).$$

the upper or lower signs being taken as n is even or odd,

$$= \left(x^n \pm \frac{1}{x^n}\right) - n.\left(x^{n-2} \pm \frac{1}{x^{n-2}}\right) + n.\frac{n-1}{2}.\left(x^{n-4} \pm \frac{1}{x^{n-4}}\right) - \ldots (2).$$

1. Let n be even;

then $(\sqrt{-1})^n = (\sqrt{-1})^{2\cdot\frac{n}{2}} = \{(\sqrt{-1})^2\}^{\frac{n}{2}} = (-1)^{\frac{n}{2}}$;

and the middle, or $\left(\frac{n}{2}+1\right)^{\text{th}}$, term of the expansion (1) is,

$$\pm n.\frac{n-1}{2}\ldots\ldots\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n};$$

+ or − according as this middle, or $\left(\frac{n}{2}+1\right)^{\text{th}}$, term is an odd or even term, *i. e.* as $\frac{n}{2}$ is even or odd. Wherefore it is of the same sign as $(-1)^{\frac{n}{2}}$; and therefore when n is even, the series (2) becomes

$$(-1)^{\frac{n}{2}}.2^n.(\sin\theta)^n = 2(\cos n\theta) - n.2\cos(n-2)\theta$$

$$+ n.\frac{n-1}{2}.2\cos(n-4)\theta - \ldots$$

$$\ldots + (-1)^{\frac{n}{2}} n.\frac{n-1}{2}\ldots\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n};$$

$$\therefore\ (-1)^{\frac{n}{2}}2^{n-1}.(\sin\theta)^n = \cos n\theta - n.\cos(n-2)\theta$$

$$+ n.\frac{n-1}{2}.\cos(n-4)\theta - \ldots$$

$$\ldots + (-1)^{\frac{n}{2}}\tfrac{1}{2}n.\frac{n-1}{2}\ldots\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n}.$$

2. If n be odd, then $(\sqrt{-1})^n = \sqrt{-1}.(\sqrt{-1})^{n-1}$

$$= (-1)^{\frac{n-1}{2}}.\sqrt{-1};$$

and the last binomial in (2) is

$$\pm n.\frac{n-1}{2}\ldots\ldots\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}.\left(x-\frac{1}{x}\right).$$

+ or − according as there is an odd or an even number of binomials in (2), i. e. as $\frac{1}{2}(n+1)$ is odd or even; the sign thereforeof this term is the same as that of $(-1)^{\frac{n+1}{2}-1}$, or $(-1)^{\frac{n-1}{2}}$, —and therefore, when n is an odd number, (2) becomes,

$$(-1)^{\frac{n-1}{2}}2^n.\sqrt{-1}.(\sin\theta)^n$$

$$= 2\sqrt{-1}.\sin n\theta - n.2\sqrt{-1}.\sin(n-2)\theta + \ldots$$

$$\ldots + (-1)^{\frac{n-1}{2}}n.\frac{n-1}{2}\ldots\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}.2\sqrt{-1}.\sin\theta;$$

$$\therefore\ (-1)^{\frac{n-1}{2}}.2^{n-1}(\sin\theta)^n$$

$$= \sin n\theta - n.\sin(n-2)\theta + n.\frac{n-1}{2}.\sin(n-4)\theta + \ldots$$

$$\ldots + (-1)^{\frac{n-1}{2}}n.\frac{n-1}{2}\ldots\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}.\sin\theta.$$

110. *To expand* $\cos n\theta$ *in terms of* $\cos\theta$ *alone*, n *being a positive integer.*

Let $2\cos\theta = a + \frac{1}{a}$; $\therefore\ 2\cos n\theta = a^n + \frac{1}{a^n}$.

$$\text{Now,}\quad (1 - ax).\left(1 - \frac{x}{a}\right) = 1 - \left(a + \frac{1}{a}\right)x + x^2$$

$$= 1 - x(b - x),$$

$$\text{if}\quad b = a + \frac{1}{a} = 2\cos\theta;$$

$$\text{Then}\quad \mathrm{l}_e(1 - ax) + \mathrm{l}_e\left(1 - \frac{x}{a}\right) = \mathrm{l}_e\{1 - x(b - x)\}.$$

Expanding these logarithms, Appendix I. 14. iii, and changing the signs of both sides of the equation, we have

$$\left.\begin{array}{l} ax + \frac{1}{2}a^2x^2 + \ldots + \frac{1}{n}.a^nx^n + \ldots \\ + \frac{x}{a} + \frac{1}{2}\frac{x^2}{a^2} + \ldots + \frac{1}{n}.x^n \quad + \ldots \end{array}\right\} = x.(b - x) + \frac{x^2}{2}.(b - x)^2 + \ldots$$

$$+ \frac{x^{n-2}}{n-2}.(b - x)^{n-2} + \frac{x^{n-1}}{n-1}.(b - x)^{n-1} + \frac{x^n}{n}.(b - x)^n + \ldots$$

And equating the coefficients of x^n,

$$\frac{1}{n}.\left(a^n + \frac{1}{a^n}\right)$$

$$= \frac{1}{n}b^n,\ \left\{\text{which is the first coefficient of } \frac{x^n}{n}.(b - x)^n\right\},$$

$$- \frac{1}{n-1}.(n-1)\,b^{n-2},\ \left\{\text{the second coefficient of } \frac{x^{n-1}}{n-1}.(b - x)^{n-1}\right\},$$

$$+ \frac{1}{n-2}.\frac{(n-2).(n-3)}{1.2}b^{n-4},\ \left\{\text{the third coefficient of } \frac{x^{n-2}}{n-2}.(b - x)^{n-2}\right\},$$

$$- \ldots\ldots\ldots\ldots$$

The general term being the coefficient of that term of

$$\frac{x^{n-r}}{n-r}.(b - x)^{n-r},$$

which involves x^n;—and this being that term in the expansion

$(b-x)^{n-r}$ which involves x^r, must be the $(r+1)^{\text{th}}$ in that expansion;—also this term is $+$ or $-$ as r is even or odd;

Therefore the general term is

$$(-1)^r \cdot \frac{1}{n-1} \cdot \frac{(n-r)\,.\,(n-r-1)\;...\;(n-2r+1)}{1\,.\,2\,.\,3...r} \cdot b^{n-2r},$$

$$\text{Or,} \quad (-1)^r \cdot \frac{(n-r-1)......(n-2r+1)}{1\,.\,2\,.\,3...r} \cdot b^{n-2r}.$$

Writing $2\cos n\theta$ for $a^n + \dfrac{1}{a^n}$, and $2\cos\theta$ for b,

the equation becomes, when both sides are multiplied by n,

$$2\cos n\theta = (2\cos\theta)^n - n\,.\,(2\cos\theta)^{n-2} + n \cdot \frac{n-3}{1\,.\,2} \cdot (2\cos\theta)^{n-4} - ...$$

[the general term being

$$(-1)^r \,.\, n \cdot \frac{(n-r-1)...(n-2r+1)}{1\,.\,2\,.\,3...r} \cdot (2\cos\theta)^{n-2r}.]$$

In determining the series from the general term by giving different values to r, r must be taken of every positive integral value not greater than $\dfrac{n}{2}$; for since none but positive integral powers of $(b-x)$ appear, $n-2r$, the index of b in the general term, must be a positive integer, or r must not be greater than $\dfrac{n}{2}$.

Note. In the proof of this Article the expansion of $l_\epsilon(1+x)$ is used, in which it might be shewn that none but positive and integral powers of x can enter. This method of proof therefore can only be applied to those cases in which n is a positive integer. In fact, it can be *demonstrated* that $\cos n\theta$ *cannot* be expressed in a series of this form when n is either negative or fractional.

111. *To express* $\cos n\theta$, n *being a positive integer, in a series ascending by integral powers of* $\cos\theta$.

In the last Article $\cos n\theta$ has been expressed in a series of *descending* integral powers of $\cos\theta$: by means of the general term there found, $\cos n\theta$ may be expressed in a series of *ascending* integral powers of $\cos\theta$.

1. Let n be even.

Since the limitation of the value of r is that it shall not be greater than $\frac{n}{2}$, for r write $\frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \ldots 3, 2, 1, 0$ successively.

The number of terms is $\frac{1}{2}n+1$; and since r is diminished successively by 1, the terms are alternately positive and negative: the first term being of the same sign as $(-1)^{\frac{n}{2}}$;

$$\therefore\ 2\cos n\theta$$

$$\begin{aligned}
&=(-1)^{\frac{n}{2}}\left\{n.\frac{(n-\frac{1}{2}n-1).(n-\frac{1}{2}n-2)\ldots 3.2.1}{1.2\ldots\frac{1}{2}n}.(2\cos\theta)^0\right.\\
&-n.\frac{\{n-(\frac{1}{2}n-1)-1\}.\{n-(\frac{1}{2}n-1)-2\}\ldots\{n-2(\frac{1}{2}n-1)+1\}}{1.2.3\ldots(\frac{1}{2}n-1)}.(2\cos\theta)^2\\
&\left.+n.\frac{\{n-(\frac{1}{2}n-2)-1\}.\{n-(\frac{1}{2}n-2)-2\}\ldots\{n-2(\frac{1}{2}n-2)+1\}}{1.2.3\ldots(\frac{1}{2}n-2)}.(2\cos\theta)^4-\ldots\right\}
\end{aligned}$$

$$\begin{aligned}
&=(-1)^{\frac{n}{2}}\left\{\frac{n.(\frac{1}{2}n-1).(\frac{1}{2}n-2)\ldots 3.2.1}{1.2\ldots\frac{1}{2}n}\right.\\
&-\frac{n.\frac{1}{2}n.(\frac{1}{2}n-1)\ldots 4.3}{1.2\ldots(\frac{1}{2}n-1)}.(2\cos\theta)^2\\
&\left.+\frac{n.(\frac{1}{2}n+1).\frac{1}{2}n.(\frac{1}{2}n-1)\ldots 6.5}{1.2.3\ldots(\frac{1}{2}n-2)}.(2\cos\theta)^4-\ldots\right\}
\end{aligned}$$

$$\begin{aligned}
&=(-1)^{\frac{n}{2}}\left\{2-\frac{n.\frac{1}{2}n}{1.2}.(2\cos\theta)^2\right.\\
&\left.+\frac{n.(\frac{1}{2}n+1).\frac{1}{2}n.(\frac{1}{2}n-1)}{1.2.3.4}.(2\cos\theta)^4-\ldots\right\},
\end{aligned}$$

and dividing by 2;

$$\operatorname{Cos} n\theta = (-1)^{\frac{n}{2}}\left\{1 - \frac{n^2}{1.2}.(\cos\theta)^2 + \frac{n^2.(n^2-2^2)}{1.2.3.4}.(\cos\theta)^4 - \frac{n^2.(n^2-2^2).(n^2-4^2)}{1.2.3.4.5.6}.(\cos\theta)^6 + \ldots\right\}.$$

2. Let n be odd; r must not be greater than $\frac{1}{2}n$; it may therefore be $\frac{1}{2}(n-1)$, the integer next less than $\frac{1}{2}n$. The number of terms is $\frac{1}{2}(n+1)$, and the terms are alternately positive and negative.

For r writing $\frac{1}{2}(n-1)$, $\frac{1}{2}(n-3)$, $\frac{1}{2}(n-5)\ldots 3, 2, 1, 0$, successively, we have as before,

$$2\cos n\theta =$$

$$(-1)^{\frac{n-1}{2}}\left\{\frac{n.\{n-\frac{1}{2}(n-1)-1\}.\{n-\frac{1}{2}(n-1)-2\}\ldots 4.3.2}{1.2.3\ldots\frac{1}{2}(n-1)}.(2\cos\theta)^1\right.$$

$$-\frac{n.\{n-\frac{1}{2}(n-3)-1\}\ldots 6.5.4}{1.2.3\ldots\frac{1}{2}(n-3)}.(2\cos\theta)^3$$

$$\left.+\frac{n.\{n-\frac{1}{2}(n-5)-1\}\ldots 7.6}{1.2.3\ldots\frac{1}{2}(n-5)}.(2\cos\theta)^5\ldots\right\}.$$

And reducing the terms in the same manner as in the last case, when n was even, we have

$$\operatorname{Cos} n\theta =$$

$$(-1)^{\frac{n-1}{2}}\left\{n.\cos\theta - \frac{n.(n^2-1^2)}{1.2.3}.(\cos\theta)^3 + \frac{n.(n^2-1^2).(n^2-3^2)}{1.2.3.4.5}.(\cos\theta)^5\right.$$

$$\left.-\frac{n.(n^2-1^2).(n^2-3^2).(n^2-5^2)}{1.2.3.4.5.6.7}.(\cos\theta)^7 + \ldots\right\}.$$

112. *Having given* $\tan\theta$, *to find* $\tan n\theta$.

$$\operatorname{Cos} n\theta + \sqrt{-1}\sin n\theta$$

$$= (\cos\theta + \sqrt{-1}\sin\theta)^n$$

$$= (\cos\theta)^n . (1 + \sqrt{-1} . \tan\theta)^n$$

$$= (\cos\theta)^n . \Big\{1 + n . \sqrt{-1} . \tan\theta - n . \frac{n-1}{2} . (\tan\theta)^2$$

$$- n . \frac{n-1}{2} . \frac{n-2}{3} . \sqrt{-1} . (\tan\theta)^3 + \ldots\Big\} \quad (1.)$$

And equating the possible and impossible parts of this equation,

$$\operatorname{Cos} n\theta = (\cos\theta)^n . \Big\{1 - n . \frac{n-1}{2} . (\tan\theta)^2$$

$$+ n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} . (\tan\theta)^4 - \ldots\Big\} \quad (2.)$$

$$\operatorname{Sin} n\theta = (\cos\theta)^n . \Big\{n\tan\theta - n . \frac{n-1}{2} . \frac{n-2}{3} . (\tan\theta)^3 + \ldots\Big\} \quad (3.)$$

$$\therefore \operatorname{Tan} n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{n . \tan\theta - n . \dfrac{n-1}{2} . \dfrac{n-2}{3} . (\tan\theta)^3 + \ldots}{1 - n . \dfrac{n-1}{2} . (\tan\theta)^2 + \ldots} .$$

Cor. If n be a positive integer, the series (1) will terminate, and the last term is $(\sqrt{-1}\tan\theta)^n$.

1. Let n be even; then

$$(\sqrt{-1}\tan\theta)^n = (\sqrt{-1})^{2 . \frac{n}{2}} (\tan\theta)^n = (-1)^{\frac{n}{2}} . (\tan\theta)^n ;$$

therefore the last term is $(-1)^{\frac{n}{2}}(\tan\theta)^n$, and

$$\text{Tan}\, n\theta = \frac{n\tan\theta - n.\dfrac{n-1}{2}.\dfrac{n-2}{3}(\tan\theta)^3 + \ldots + (-1)^{\frac{n}{2}-1}n(\tan\theta)^{n-1}}{1 - n.\dfrac{n-1}{2}(\tan\theta)^2 \ldots\ldots + (-1)^{\frac{n}{2}}(\tan\theta)^n}.$$

2. Let n be odd; then

$$(\sqrt{-1}\tan\theta)^n = \sqrt{-1}.(\sqrt{-1})^{n-1}.(\tan\theta)^n = \sqrt{-1}.(-1)^{\frac{n-1}{2}}(\tan\theta)^n;$$

therefore the last term is $(-1)^{\frac{n-1}{2}}(\tan\theta)^n$, and

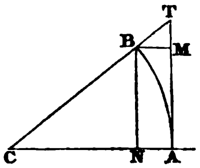
$$\text{Tan}\, n\,\theta = \frac{n.\tan\theta - n.\dfrac{n-1}{2}.\dfrac{n-2}{3}(\tan\theta)^3 + \ldots + (-1)^{\frac{n-1}{2}}(\tan\theta)^n}{1 - n.\dfrac{n-1}{2}(\tan\theta)^2 + \ldots\ldots + (-1)^{\frac{n-1}{2}}n(\tan\theta)^{n-1}}.$$

[The sign of $n(\tan\theta)^{n-1}$ is the same as that of $(\tan\theta)^n$, because every *odd* term in the expansion (1) has the same sign as the term immediately following it; and when n is odd, $n(\tan\theta)^{n-1}$ is an even term, and has therefore the same sign as $(\tan\theta)^n$.]

113. *As the angle, (which is supposed to be less at first than a right angle), is continually diminished, each of the quantities* $\dfrac{\sin\theta}{\theta}$ *and* $\dfrac{\tan\theta}{\theta}$ *approaches to* 1, *and when* θ *vanishes, each of them* does *become* 1.

Let AB be the arc of a circle whose centre is C and radius AC.

Let $\angle ACB = \theta$; and let AT touch the circle at A.



From B draw BN perpendicular, and BM parallel, to AC.

Then MN is a rectangle, and

$$AM = NB, \quad AN = MB.$$

Now arc $AB >$ the straight line joining A and B, and $\therefore > NB$; and arc $AB < AM + MB$; (*Legendre's Geometry;* IV. 9.)

$$\therefore \quad \frac{AB}{AC} > \frac{NB}{AC} \text{ and } < \frac{NB}{AC} + \frac{AN}{AC};$$

$$\text{or,} \quad \theta > \sin\theta \text{ and } < \sin\theta + \text{versin}\,\theta;$$

$$\therefore \quad \frac{\theta}{\sin\theta} > 1 \text{ and } < 1 + \frac{1-\cos\theta}{\sin\theta},$$

$$\text{And} \quad \frac{1-\cos\theta}{\sin\theta} = \frac{2\left(\sin\frac{\theta}{2}\right)^2}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2},$$

$$\therefore \quad \frac{\theta}{\sin\theta} > 1 \text{ and } < 1 + \tan\frac{\theta}{2}.$$

Now if θ be continually diminished till it at last vanishes, $\tan\frac{\theta}{2}$ continually decreases, and at last vanishes, in which case the value of $\frac{\theta}{\sin\theta}$, and therefore that of $\frac{\sin\theta}{\theta}$, becomes 1.

Also; $\frac{\tan\theta}{\theta} = \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta}$, and if θ vanishes, this quantity must become 1, since when the angle vanishes the value of the cosine is 1.

114. *The area of a circle of radius* r *is* πr^2.

The area of a regular polygon of n sides inscribed in the circle is n times the area of a triangle two of whose sides are r, r and the included angle $\frac{2\pi}{n}$;

$$\therefore \text{ Area of the polygon} = n \cdot \frac{r^2}{2} \cdot \sin\frac{2\pi}{n} = \pi r^2 \cdot \frac{\sin\frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)}.$$

Now as the number of the sides increases, the area of the polygon approaches nearer and nearer to that of the circle, and when n is infinite becomes identical with it,—in which case $\frac{2\pi}{n}$ becomes an infinitely small angle, and therefore $\frac{\sin\frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)}$ becomes 1. Art. 113.

$$\therefore \text{ Area of the circle} = \pi r^2.$$

115. *To shew that* $\text{Cos}\, a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \ldots$

and $\text{Sin}\, a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \ldots$

Since, Art. 112. (2),

$$\text{Cos}\, n\theta = (\cos\theta)^n \cdot \left\{1 - n \cdot \frac{n-1}{2} \cdot (\tan\theta)^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot (\tan\theta)^4 - \ldots\right\},$$

$$\text{let } n = \frac{a}{\theta};$$

$$\therefore\ \text{Cos}\,\alpha = (\cos\theta)^n.\left\{1 - \frac{\alpha}{\theta}.\frac{\frac{\alpha}{\theta}-1}{2}\ (\tan\theta)^2\right.$$

$$\left. + \frac{\alpha}{\theta}.\frac{\frac{\alpha}{\theta}-1}{2}.\frac{\frac{\alpha}{\theta}-2}{3}.\frac{\frac{\alpha}{\theta}-3}{4}.(\tan\theta)^4 - \ldots\right\}$$

$$=(\cos\theta)^n.\left\{1-\alpha.\frac{\alpha-\theta}{2}.\left(\frac{\tan\theta}{\theta}\right)^2+\alpha.\frac{\alpha-\theta}{2}.\frac{\alpha-2\theta}{3}.\frac{\alpha-3\theta}{4}.\left(\frac{\tan\theta}{\theta}\right)^4-\ldots\right\}.$$

Now this is true whatever be the value that is given to θ; and if 0 be written for θ,—in which case $\frac{\tan\theta}{\theta} = 1$, and $\cos\theta = 1$,—we have

$$\therefore\ \text{Cos}\,\alpha = 1 - \frac{\alpha^2}{1.2} + \frac{\alpha^4}{1.2.3.4} - \ldots\ldots\ldots\ldots\ldots\ (1.)$$

By a similar substitution in the expression of Art. 112, (3),

$$\text{Sin}\,n\theta = (\cos\theta)^n.\left\{n\tan\theta - n.\frac{n-1}{2}.\frac{n-2}{3}.(\tan\theta)^3 + \ldots\right\},$$

we obtain

$$\text{Sin}\,\alpha = \alpha - \frac{\alpha^3}{1.2.3} + \frac{\alpha^5}{1.2.3.4.5} - \ldots\ldots\ldots\ldots\ldots\ (2.)$$

If α be less than $\frac{\pi}{2}$, or a right angle, the series (1) and (2) are immediately convergent.

Note. To arrive at these expressions we have supposed $\frac{\tan\theta}{\theta}$ to become 1 when 0 is written for θ, which is the case only when the angle is referred to the *circular* measure $\frac{\text{arc}}{\text{radius}}$. Hence α, which is equal to $n\theta$, is also referred to the same measure in the series (1) and (2).

116. *The sine of a very small angle is equal to the angle itself very nearly,—the angle being expressed by the measure* $\dfrac{\text{arc}}{\text{radius}}$.

Let a be a very small angle.

Then, by the last Article,

$$\operatorname{Sin} a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \ldots = a.\left(1 - \frac{a^2}{1.2.3} + \ldots\right).$$

Now a being a very small angle, a^2 and all higher powers of a may be neglected when compared with 1;

$$\therefore \operatorname{Sin} a = a \text{ nearly, when } a \text{ is very small.}$$

Cor. If a and β be two *very small* angles, we have

$$\frac{\operatorname{Sin} a}{\operatorname{Sin} \beta} = \frac{a}{\beta} = \frac{\text{number of seconds in } \angle a}{\text{number of seconds in } \angle \beta}.$$

Hence, $\dfrac{\sin 2''}{\sin 1''} = \dfrac{2}{1}$; $\dfrac{\sin 3''}{\sin 1''} = \dfrac{3}{1}$; and generally, if n be any small number,

$$\frac{\sin n''}{\sin 1''} = \frac{n}{1};$$

$$\therefore \operatorname{Sin} 2'' = 2 \sin 1'', \quad \sin 3'' = 3 \sin 1'', \quad \sin n'' = n \sin 1''.$$

117. If a be an angle so small that a^2 and higher powers of a may be neglected when compared with unity, equation (1) of Art. 115, becomes

$$\operatorname{Cos} a = 1.$$

If a^2, a^3 be retained, but higher powers of a be neglected, we have from (1) and (2),

$$\operatorname{Sin} a = a - \frac{a^3}{6}; \qquad \operatorname{Cos} a = 1 - \frac{a^2}{2}.$$

Note. These approximations are often practically very useful.

118. *If* $\epsilon^{\theta\sqrt{-1}}$ *and* $\epsilon^{-\theta\sqrt{-1}}$ *be expanded in terms of* $\theta\sqrt{-1}$ and $-\theta\sqrt{-1}$ *in the same manner as* ϵ^{x} *is expanded in terms of* x *in the series,*

$$\epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots \text{ App. I. 18. Cor.}$$

then, (1). $\operatorname{Cos}\theta = \frac{1}{2}.(\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}})$

(2). $\operatorname{Sin}\theta = \dfrac{1}{2\sqrt{-1}}.(\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}})$

(3). $\operatorname{Tan}\theta = \dfrac{1}{\sqrt{-1}}.\dfrac{\epsilon^{2\theta\sqrt{-1}} - 1}{\epsilon^{2\theta\sqrt{-1}} + 1}.$

For, $\epsilon^{\theta\sqrt{-1}} = 1 + \theta.\sqrt{-1} - \dfrac{\theta^2}{1.2} - \dfrac{\theta^3}{1.2.3}.\sqrt{-1} + \dfrac{\theta^4}{1.2.3.4} + \ldots$

and $\epsilon^{-\theta\sqrt{-1}} = 1 - \theta.\sqrt{-1} - \dfrac{\theta^2}{1.2} + \dfrac{\theta^3}{1.2.3}.\sqrt{-1} + \dfrac{\theta^4}{1.2.3.4} - \ldots$

$\because\ \epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}} = 2.(1 - \dfrac{\theta^2}{1.2} + \dfrac{\theta^4}{1.2.3.4} - \ldots) = 2\cos\theta$; Art. 115.

and $\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}} = 2.\sqrt{-1}.(\theta - \dfrac{\theta^3}{1.2.3} + \ldots) = 2\sqrt{-1}.\sin\theta$;

$\therefore\ \operatorname{Cos}\theta = \frac{1}{2}(\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}})\ldots\ldots\ldots\ldots\ldots(1.),$

And, $\operatorname{Sin}\theta = \dfrac{1}{2\sqrt{-1}}(\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}})\ \ldots\ldots\ldots\ (2.);$

Also, $\operatorname{Tan}\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\sqrt{-1}}.\dfrac{\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}}{\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\sqrt{-1}}};$

and multiplying the numerator and the denominator by $\epsilon^{\theta\sqrt{-1}}$,

$$\operatorname{Tan}\theta = \frac{1}{\sqrt{-1}}.\frac{\epsilon^{2\theta\sqrt{-1}} - 1}{\epsilon^{2\theta\sqrt{-1}} + 1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3.)$$

119. *To prove Gregorie's series, which is*

$$\theta = \tan\theta - \frac{(\tan\theta)^3}{3} + \frac{(\tan\theta)^5}{5} - \dots$$

$$\epsilon^{\theta\sqrt{-1}} + \epsilon^{-\theta\sqrt{-1}} = 2\cos\theta,$$

and $\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}} = 2\sqrt{-1}\,.\,\sin\theta$;

$$\therefore\ \epsilon^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\,.\,\sin\theta;$$

$$\therefore\ \epsilon^{\theta\sqrt{-1}} = \cos\theta\,.\,(1 + \sqrt{-1}\tan\theta);$$

$$\therefore\ \mathrm{l}_\epsilon \epsilon^{\theta\sqrt{-1}} = \mathrm{l}_\epsilon \cos\theta + \mathrm{l}_\epsilon(1 + \sqrt{-1}\tan\theta)$$

$$\theta\sqrt{-1} = \mathrm{l}_\epsilon\cos\theta + \left\{\sqrt{-1}\tan\theta - \frac{(\sqrt{-1}\tan\theta)^2}{2} + \frac{(\sqrt{-1}\tan\theta)^2}{3} - \dots\right\}$$

(App. I. 14. iii.)

$$= \mathrm{l}_\epsilon\cos\theta + \sqrt{-1}\tan\theta + \frac{(\tan\theta)^2}{2} - \sqrt{-1}\,\frac{(\tan\theta)^3}{3} - \dots$$

and equating the possible and impossible parts of this equation, we have

$$0 = \mathrm{l}_\epsilon\cos\theta + \frac{(\tan\theta)^2}{2} - \frac{(\tan\theta)^4}{4} + \frac{(\tan\theta)^6}{6} - \dots$$

and $\theta = \tan\theta - \frac{(\tan\theta)^3}{3} + \frac{(\tan\theta)^5}{5} - \dots$

If $\tan\theta$ be not greater than 1, or θ be any angle not greater than half a right angle, this series is convergent.

Cor. Tangent of $\frac{1}{2}$ a right angle, $\left(\text{or } \tan\frac{\pi}{4}\right)$, $=1$;

$$\therefore\ \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\dots$$

$$=\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\dots;$$

$$=2\left(\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\dots\right),$$

a series which is not convergent enough to enable us to calculate the value of π very easily.

120. *To prove Euler's series for determining the value of π, which is*

$$\frac{\pi}{4}=\left(\frac{1}{2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\dots\right)$$

$$+\left(\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\dots\right).$$

By Art. 67,

$$\text{Tan}^{-1}1-\tan^{-1}\tfrac{1}{2}=\tan^{-1}\frac{1-\frac{1}{2}}{1+\frac{1}{2}}=\tan^{-1}\tfrac{1}{3};$$

$$\therefore\ \tan^{-1}1=\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3},$$

$$\text{and } \tan^{-1}\tfrac{1}{2}=\frac{1}{2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\dots \quad \text{Art. 119.}$$

$$\tan^{-1}\tfrac{1}{3}=\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\dots;$$

$$\therefore\ \text{Tan}^{-1}1,\ \text{or } \frac{\pi}{4},\ =\left(\frac{1}{2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\dots\right)$$

$$+\left(\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\dots\right),$$

a series which converges much more rapidly than that deduced in the Corollary to the last Article.

121. *To prove Machin's series for determining* π, *which is*

$$\frac{\pi}{4}=4\left(\frac{1}{5}-\frac{1}{3\,.\,5^3}+\frac{1}{5\,.\,5^5}-\dots\right)$$
$$-\left(\frac{1}{239}-\frac{1}{3\,.\,(239)^3}+\frac{1}{5\,.\,(239)^5}-\dots\right).$$

By Art. 67, $\text{Tan}^{-1}1-\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{1-\frac{1}{5}}{1+\frac{1}{5}}=\tan^{-1}\frac{2}{3}$,

$$\tan^{-1}\frac{2}{3}-\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{\frac{2}{3}-\frac{1}{5}}{1+\frac{2}{3\,.\,5}}=\tan^{-1}\frac{7}{17},$$

$$\tan^{-1}\frac{7}{17}-\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{\frac{7}{17}-\frac{1}{5}}{1+\frac{7}{5\,.\,17}}=\tan^{-1}\frac{9}{46},$$

$$\tan^{-1}\frac{9}{46}-\tan^{-1}\frac{1}{5}=\tan^{-1}\frac{\frac{9}{46}-\frac{1}{5}}{1+\frac{9}{5\,.\,46}}=\tan^{-1}\left(\frac{-1}{239}\right);$$

adding these equations together, and omitting the terms common to each side of the resulting equation, we have

$$\text{Tan}^{-1}1-4\tan^{-1}\frac{1}{5}=\tan^{-1}\left(\frac{-1}{239}\right);$$

$$\therefore\ \tan^{-1}1=4\tan^{-1}\frac{1}{5}+\tan^{-1}\left(\frac{-1}{239}\right);$$

$$\therefore\ \frac{\pi}{4}=4\left(\frac{1}{5}-\frac{1}{3\,.\,5^3}+\frac{1}{5\,.\,5^5}-\dots\right)$$
$$-\left\{\frac{1}{239}-\frac{1}{3\,.\,(239)^3}+\frac{1}{5\,.\,(239)^5}-\dots\right\}.$$

In this way it is found that

$$\pi=3\,.\,141592653589793\dots$$

122. *The tangent of a very small angle is equal to the angle itself very nearly; the angle being expressed by the measure* $\frac{\text{arc}}{\text{radius}}$.

For $\theta = \tan\theta - \frac{(\tan\theta)^3}{3} + \ldots$ Art. 119. $= \tan\theta . \left\{1 - \frac{(\tan\theta)^2}{3} + \ldots\right\}$.

Now when θ is very small, $\tan\theta$ is very small, and therefore $(\tan\theta)^2$, and all higher powers of $\tan\theta$, may be neglected when compared with 1;

$$\therefore\ \theta = \tan\theta, \text{ nearly, when } \theta \text{ is very small.}$$

Cor. Hence, if α and β be two *very small* angles, we have

$$\frac{\operatorname{Tan}\alpha}{\operatorname{Tan}\beta} = \frac{\alpha}{\beta} = \frac{\text{number of seconds in } \angle\alpha}{\text{number of seconds in } \angle\beta}.$$

123. *Having given* $\tan\alpha$, $\tan\beta$, ... $\tan\lambda$, *to find* $\tan(\alpha + \beta + \ldots + \lambda)$.

$$(\operatorname{Cos}\alpha + \sqrt{-1}\sin\alpha) . (\cos\beta + \sqrt{-1}\sin\beta)$$

$$= (\cos\alpha . \cos\beta - \sin\alpha . \sin\beta) + \sqrt{-1} . (\cos\alpha . \sin\beta + \cos\beta . \sin\alpha)$$

$$= \cos(\alpha + \beta) + \sqrt{-1} . \sin(\alpha + \beta) \ldots\ldots\ldots\ldots (1).$$

Suppose this law to hold for n factors, so that

$$(\cos\alpha + \sqrt{-1} . \sin\alpha) . (\cos\beta + \sqrt{-1} . \sin\beta) \ldots (\cos\kappa + \sqrt{-1} . \sin\kappa)$$

$$= \cos(\alpha + \beta + \ldots + \kappa) + \sqrt{-1} . \sin(\alpha + \beta + \ldots + \kappa);$$

then by introducing another factor, we have

$$(\cos\alpha + \sqrt{-1} . \sin\alpha) \ldots (\cos\kappa + \sqrt{-1} . \sin\kappa) . (\cos\lambda + \sqrt{-1} . \sin\lambda)$$

$$= \left\{\cos(\alpha + \beta + \ldots + \kappa) + \sqrt{-1} . \sin(\alpha + \beta + \ldots + \kappa)\right\} . (\cos\lambda + \sqrt{-1} . \sin\lambda)$$

$$= \cos(\alpha + \beta + \ldots + \lambda) + \sqrt{-1} . \sin(\alpha + \beta + \ldots + \lambda), \text{ by (1)};$$

and therefore the law holds for $n + 1$ factors.

But by actual multiplication the law has been shewn to hold for two factors, it must therefore hold for three factors, and thus by successive inductions we conclude that it holds generally.

Let S_1 = the sum of the quantities $\tan\alpha$, $\tan\beta$, ...

S_2 = the sum of the products of every two of them,

S_3 = the sum of the products of every three of them; and so on.

Then $\cos(\alpha+\beta+\dots+\lambda)+\sqrt{-1}\,.\,\sin(\alpha+\beta+\dots+\lambda)$

$$= (\cos\alpha+\sqrt{-1}\,.\,\sin\alpha)\,.\,(\cos\beta+\sqrt{-1}\,.\,\sin\beta)\dots(\cos\lambda+\sqrt{-1}\,.\,\sin\lambda)$$

$$= \cos\alpha\,.\,(1+\sqrt{-1}\,.\tan\alpha)\,.\cos\beta\,.\,(1+\sqrt{-1}\,.\tan\beta)\dots\cos\lambda\,.\,(1+\sqrt{-1}\,.\tan\lambda)$$

$$= \cos\alpha\,.\,\cos\beta\dots\cos\lambda\,.\,(1+\sqrt{-1}\,.\,S_1 - S_2 - \sqrt{-1}\,.\,S_3 + S_4 + \dots).$$

Wood's Algebra, Art. 271.

Hence, equating the possible and impossible parts, we have

$$\text{Cos}(\alpha+\beta+\dots+\lambda) = \cos\alpha\,.\,\cos\beta\dots\cos\lambda\,.\,(1 - S_2 + S_4 - S_6 + \dots)$$

$$\text{Sin}(\alpha+\beta+\dots+\lambda) = \cos\alpha\,.\,\cos\beta\dots\cos\lambda\,.\,(S_1 - S_3 + S_5 - \dots);$$

$$\therefore\ \text{Tan}(\alpha+\beta+\dots+\lambda) = \frac{\sin(\alpha+\beta+\dots+\lambda)}{\cos(\alpha+\beta+\dots+\lambda)} = \frac{S_1 - S_3 + S_5 - \dots}{1 - S_2 + S_4 - \dots}.$$

If there be n of the angles α, β, ... λ, it may easily be shewn, as in Art. 112. Cor. that

$$\text{Tan}(\alpha+\beta+\dots+\lambda) = \frac{S_1 - S_3 + S_5 - \dots + (-1)^{\frac{n}{2}-1} S_{n-1}}{1 - S_2 + S_4 - \dots + (-1)^{\frac{n}{2}} S_n},\ n \text{ even}.$$

$$= \frac{S_1 - S_3 + S_5 - \dots + (-1)^{\frac{n-1}{2}} S_n}{1 - S_2 + S_4 - \dots + (-1)^{\frac{n-1}{2}} S_{n-1}},\ n \text{ odd}.$$

124. *If* sin p = sin P . sin (z + p), *required to expand* p *in terms of* sin P *and of the sines of* z *and its multiples.*

Hymers' Astronomy, Art. 233.

Since $\operatorname{Sin} p = \dfrac{1}{2\sqrt{-1}}\left(\epsilon^{p\sqrt{-1}} - \epsilon^{-p\sqrt{-1}}\right)$ Art. 118.

the equation becomes by substituting such values of Sin p and Sin $(z+p)$,

$$\frac{1}{2\sqrt{-1}} \cdot \left(\epsilon^{p\sqrt{-1}} - \epsilon^{-p\sqrt{-1}}\right) = \sin P . \frac{1}{2\sqrt{-1}} \cdot \left(\epsilon^{(z+p)\cdot\sqrt{-1}} - \epsilon^{-(z+p)\cdot\sqrt{-1}}\right)$$

and multiplying both sides of the equation by $2\sqrt{-1} . \epsilon^{p\sqrt{-1}}$, we have

$$\epsilon^{2p\sqrt{-1}} - 1 = \sin P . \left(\epsilon^{(z+2p)\sqrt{-1}} - \epsilon^{-z\sqrt{-1}}\right)$$

$$= \sin P . \epsilon^{z\sqrt{-1}} . \epsilon^{2p\sqrt{-1}} - \sin P . \epsilon^{-z\sqrt{-1}};$$

$$\therefore \epsilon^{2p\sqrt{-1}}\left(1 - \sin P . \epsilon^{z\sqrt{-1}}\right) = 1 - \sin P . \epsilon^{-z\sqrt{-1}};$$

$$\therefore \epsilon^{2p\sqrt{-1}} = \frac{1 - \sin P . \epsilon^{-z\sqrt{-1}}}{1 - \sin P . \epsilon^{z\sqrt{-1}}};$$

$$\therefore 2p\sqrt{-1} = \text{l}_\epsilon\left(1 - \sin P . \epsilon^{-z\sqrt{-1}}\right) - \text{l}_\epsilon\left(1 - \sin P . \epsilon^{z\sqrt{-1}}\right)$$

$$= -\sin P . \epsilon^{-z\sqrt{-1}} - \tfrac{1}{2}(\sin P)^2 . \epsilon^{-2z\sqrt{-1}} - \tfrac{1}{3}(\sin P)^3 . \epsilon^{-3z\sqrt{-1}} - \ldots$$

$$+ \sin P . \epsilon^{z\sqrt{-1}} + \tfrac{1}{2}(\sin P)^2 . \epsilon^{2z\sqrt{-1}} + \tfrac{1}{3}(\sin P)^3 . \epsilon^{3z\sqrt{-1}} + \ldots$$

Appendix I. 14. iii.

$$\therefore p = \sin P . \frac{1}{2\sqrt{-1}} . \left(\epsilon^{z\sqrt{-1}} - \epsilon^{-\sqrt{-1}}\right)$$

$$+ \tfrac{1}{2}(\sin P)^2 . \frac{1}{2\sqrt{-1}} . \left(\epsilon^{2z\sqrt{-1}} - \epsilon^{-2z\sqrt{-1}}\right) + \ldots$$

$$= \sin P . \sin z + \tfrac{1}{2}(\sin P)^2 . \sin 2z + \tfrac{1}{3}(\sin P)^3 . \sin 3z + \ldots$$

125. In the same manner if $\tan l' = n . \tan l$, and the tangents be expressed in terms of $l\sqrt{-1}$ and $l'\sqrt{-1}$ by the formula of Art. 118, it may be proved that,

$$l' = l - m . \sin 2l + \tfrac{1}{2}m^2 . \sin 4l - \tfrac{1}{3}m^3 . \sin 6l + \dots$$

$$\text{where} \quad m = \frac{1-n}{1+n}.$$

126. *Required the number of seconds contained in the angle θ, which is expressed by the measure $\frac{arc}{radius}$.*

Let a be the required number of seconds.

Now in the same circle

$$\frac{\text{arc subtending } a''}{\text{arc subtending } 1''} = \frac{a}{1},$$

$$\therefore a = \frac{\dfrac{\text{arc subtending } a''}{\text{radius}}}{\dfrac{\text{arc subtending } 1''}{\text{radius}}}.$$

But by hypothesis, $\dfrac{\text{arc subtending } a''}{\text{radius}} = \theta$,

and since $1''$ is a very small angle,

$$\therefore \frac{\text{arc subtending } 1''}{\text{radius}} = \sin 1'', \text{ nearly,} \quad \text{Art. 116;}$$

$$\therefore a = \frac{\theta}{\sin 1''}.$$

NOTE. *This result is often of great use.*

Ex. 1. Thus the number of seconds contained in the angle p, which, Art. 124, is expressed by the measure $\frac{\text{arc}}{\text{radius}}$, is

$$\frac{p}{\sin 1''} = \frac{\sin P}{\sin 1''} . \sin x + \tfrac{1}{2} . \frac{(\sin P)^2}{\sin 1''} . \sin 2x + \tfrac{1}{3} . \frac{(\sin P)^3}{\sin 1''} . \sin 3x + \dots$$

$$\text{or } = \frac{\sin P}{\sin 1''} . \sin x + \frac{(\sin P)^2}{\sin 2''} \sin 2x + \frac{(\sin P)^3}{\sin 3''} . \sin 3x + \dots \text{Art. 117.}$$

Ex. 2. In Art. 125 we have

The number of seconds in l'

$$= \frac{l'}{\sin 1''} = \frac{l}{\sin 1''} - \frac{m}{\sin 1''} . \sin 2l + \tfrac{1}{2} . \frac{m^2}{\sin 1''} . \sin 4l - \dots$$

$$\text{or } = \frac{l}{\sin 1''} - \frac{m}{\sin 1''} . \sin 2l + \frac{m^2 . \sin 4l}{\sin 2''} - \dots$$

[The angle l, which is the observed latitude of a place, is read off in degrees, minutes, and seconds from the instrument by which the observation is made; therefore to find the degrees, minutes, and seconds in l', we have only to determine the number of seconds in the latter part of the series, viz.

$$- \frac{m}{\sin 1''} . \sin 2l + \frac{m^2}{\sin 2''} . \sin 4l - \dots].$$

127. *To expand* $\frac{1}{1 - \mathrm{e} \cos \theta}$ *in a series of the cosines of* θ *and its multiples;* e *being less than* 1.

By hypothesis, e is less than 1; and since $(1 - b)^2$, or $1 - 2b + b^2$, being a square, is necessarily positive, $1 + b^2$ is greater than $2b$. Let therefore

$$e = \frac{2b}{1 + b^2};$$

$$\therefore\ 1 - e^2 = 1 - \frac{4b^2}{(1 + b^2)^2} = \left(\frac{1 - b^2}{1 + b^2}\right)^2,$$

$$\therefore\ \frac{1 - b^2}{1 + b^2} = \sqrt{1 - e^2};$$

Whence, $b^2 = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} \cdot \frac{1+\sqrt{1-e^2}}{1+\sqrt{1-e^2}}$;

$$\therefore\ b = \frac{e}{1+\sqrt{1-e^2}}.$$

Let $2\cos\theta = x + \frac{1}{x}$;

$$\therefore\ 1 - e\cos\theta = 1 - \frac{b}{1+b^2}\cdot\left(x+\frac{1}{x}\right)$$

$$= \frac{1}{1+b^2}\cdot\left\{1 - b\cdot\left(x+\frac{1}{x}\right) + b^2\right\}$$

$$= \frac{1}{1+b^2}\cdot(1-bx)\cdot\left(1-\frac{b}{x}\right);$$

$$\therefore\ \frac{1}{1-e\cos\theta} = (1+b^2)\cdot\frac{1}{1-bx}\cdot\frac{1}{1-\frac{b}{x}}$$

$$= (1+b^2)\cdot\left\{\begin{array}{c}(1 + bx + b^2x^2 + b^3x^3 + \ldots)\\ \times\left(1 + \frac{b}{x} + \frac{b^2}{x^2} + \frac{b^3}{x^3} + \ldots\right)\end{array}\right\}$$

$$\begin{aligned} = (1+b^2)\cdot\Big\{\, & 1 + b^2 + b^4 + \ldots \\ & + (b + b^3 + b^5 + \ldots)\cdot\left(x+\frac{1}{x}\right) \\ & + (b^2 + b^4 + b^6 + \ldots)\cdot\left(x^2+\frac{1}{x^2}\right) \\ & \ldots\ldots\ldots\ldots\ldots \\ & + (b^n + b^{n+2} + b^{n+4} + \ldots)\cdot\left(x^n+\frac{1}{x^n}\right) \\ & + \ldots\ldots\ldots\ldots\ldots\Big\}. \end{aligned}$$

Now $$1 + b^2 + b^4 + \ldots = \frac{1}{1-b^2}$$

$$b + b^3 + b^5 + \ldots = \frac{b}{1-b^2}$$

$$\ldots\ldots\ldots\ldots\ldots = \ldots\ldots$$

$$b^n + b^{n+2} + b^{n+4} + \ldots\ldots = \frac{b^n}{1-b^2}$$

$$\therefore \frac{1}{1-e\cos\theta} = (1+b^2).\left\{\frac{1}{1-b^2} + \frac{b}{1-b^2}.2\cos\theta + \frac{b^2}{1-b^2}.2\cos 2\theta + \ldots\right\}$$

$$= \frac{1+b^2}{1-b^2}.\{1 + 2b.\cos\theta + 2b^2.\cos 2\theta + \ldots\}$$

$$= \frac{1}{\sqrt{1-e^2}}.\{1 + 2b.\cos\theta + 2b^2.\cos 2\theta + \ldots\}.$$

Since e is less than 1, b, or $\frac{e}{1+\sqrt{1-e^2}}$, is small, and therefore this series converges rapidly.

Cor. The equation to an ellipse is $r = a.\frac{(1-e^2)}{1-e\cos\theta}$, the focus being the pole, and θ being measured from that vertex which is the further from the focus;

$$\therefore\ r = a.\sqrt{1-e^2}.\{1 + 2b.\cos\theta + 2b^2.\cos 2\theta + \ldots\},$$

$$\text{where}\quad b = \frac{e}{1+\sqrt{1-e^2}}.$$

128. *To expand* $l_e(1 - e\cos\theta)$ *in a series of the cosines of* θ *and its multiples.*

As in the last Article,

$$1 - e\cos\theta = \frac{1}{1+b^2}.(1-bx).\left(1-\frac{b}{x}\right),$$

$$\text{where } b = \frac{e}{1+\sqrt{1-e^2}},\quad \text{and } 2\cos\theta = x + \frac{1}{x};$$

$$\therefore\ \mathrm{l}_e(1-e\cos\theta)=\mathrm{l}_e\frac{1}{1+b^2}+\mathrm{l}_e(1-bx)+\mathrm{l}_e\left(1-\frac{b}{x}\right)$$

$$=\mathrm{l}_e\frac{1}{1+b^2}\ -bx-\tfrac{1}{2}b^2x^2-\tfrac{1}{3}b^3x^3-\dots$$

$$-\frac{b}{x}-\tfrac{1}{2}\frac{b^2}{x^2}-\tfrac{1}{3}\frac{b^3}{x^3}-\dots$$

$$=\mathrm{l}_e\frac{1}{1+b^2}\ -b.2\cos\theta-\tfrac{1}{2}b^2.2\cos2\theta-\tfrac{1}{3}b^3.2\cos3\theta-\dots$$

And since $b=\dfrac{e}{1+\sqrt{1-e^2}}$,

$$\therefore\ \frac{1}{1+b^2}=\frac{(1+\sqrt{1-e^2})^2}{\{1+\sqrt{1-e^2}\}^2+e^2}=\frac{(1+\sqrt{1-e^2})^2}{2+2\sqrt{1-e^2}}=\tfrac{1}{2}.(1+\sqrt{1-e^2}).$$

129. *To expand* $(a^2-2ab\cos\theta+b^2)^n$ *in a series of the cosines of* θ *and its multiples.*

Of the two quantities a and b, let a be the greater.

Now $(a^2-2ab.\cos\theta+b^2)^n=a^{2n}\left(1-2.\dfrac{b}{a}.\cos\theta+\dfrac{b^2}{a^2}\right)^n.$

Let $2\cos\theta=x+\dfrac{1}{x}$, and $\dfrac{b}{a}=c$;

$$\therefore\ \left(1-2.\frac{b}{a}\cos\theta+\frac{b^2}{a^2}\right)$$

$$=\left\{1-c.\left(x+\frac{1}{x}\right)+c^2\right\}^n$$

$$=(1-cx)^n.\left(1-\frac{c}{x}\right)^n$$

$$=\left\{1-n.cx+n.\frac{n-1}{2}.c^2x^2-n.\frac{n-1}{2}.\frac{n-2}{3}.c^3x^3+\dots\right\}$$

$$\times\left\{1-n.\frac{c}{x}+n.\frac{n-1}{2}.\frac{c^2}{x^2}-n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{c^3}{x^3}+\dots\right\}.$$

Performing the multiplication, we have for the first term, which does not involve x,

$$1 + (nc)^2 + (n . \frac{n-1}{2} . c^2)^2 + (n . \frac{n-1}{2} . \frac{n-2}{3} . c^3)^2 + \dots$$

The coefficient of x, and also that of $\frac{1}{x}$, is

$$-nc - n . \frac{n-1}{2} . c^2 \times nc - n . \frac{n-1}{2} . \frac{n-2}{3} . c^3 \times n . \frac{n-1}{2} . c^2 - \dots$$

$$\text{or} - \left\{nc + n^2 . \frac{n-1}{2} . c^3 + \left(n . \frac{n-1}{2}\right)^2 . \frac{n-2}{3} . c^5 + \dots\dots\right\}$$

The coefficient of x^2, and also that of $\frac{1}{x^3}$, is

$$+n . \frac{n-1}{2} . c^2 + n^2 . \frac{n-1}{2} . \frac{n-2}{3} . c^4 + \left(n . \frac{n-1}{2}\right)^2 . \frac{n-2}{3} \ \frac{n-3}{4} . c^6 + \dots$$

and so on.

$$\text{Now,} \quad x + \frac{1}{x} = 2\cos\theta, \quad x^2 + \frac{1}{x^2} = 2\cos 2\theta, \ \text{\&c.};$$

wherefore we have

$$(a^2 - 2ab\cos\theta + b^2)^n$$

$$= a^{2n} . (1 - cx)^n . \left(1 - \frac{c}{x}\right)^n$$

$$= a^{2n} . \left\{1 + \left(n . \frac{b}{a}\right)^2 + \left(n . \frac{n-1}{2} . \frac{b^2}{a^2}\right)^2 + \left(n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{b^3}{a^3}\right)^2 + \dots\right.$$

$$-2\cos\theta . \left\{n . \frac{b}{a} + n^2 . \frac{n-1}{2} . \frac{b^3}{a^3} + \left(n . \frac{n-1}{2}\right)^2 . \frac{n-2}{3} . \frac{b^5}{a^5} + \dots\right\}$$

$$+ 2\cos 2\theta . \left\{n . \frac{n-1}{2} . \frac{b^2}{a^2} + n^2 . \frac{n-1}{2} . \frac{n-2}{3} . \frac{b^4}{a^4} + \dots\right\}$$

$$\left. - \dots\dots \right\}.$$

In a manner similar to that employed in Art. 128, we may expand $\text{l}_e(a^2 - 2ab\cos\theta + b^2)^n$.

130. The approximations of Art. 116, 117, to the values of the sine and the cosine of a very small angle, may often be applied to determine the magnitude of Astronomical Corrections, when they are very small quantities.

Ex. 1. *If* $\text{Sin}(\omega - \text{y}) = \sin\omega \,.\, \cos \text{u}$, *where* y *is very small, required an approximate value of* y.

Here, $\text{Sin}\,\omega \,.\, \cos y - \cos\omega \,.\, \sin y = \sin\omega \,.\, \cos u$;

$$\therefore\ \left(1 - \frac{y^2}{2}\right) . \sin\omega - y \,.\, \cos\omega = \sin\omega \,.\, \cos u;$$

$$\begin{aligned}\therefore\ y \,.\, \cos\omega + \frac{y^2}{2} \,.\, \sin\omega &= \sin\omega - \sin\omega \,.\, \cos u\\ &= \sin\omega \,.\, (1 - \cos u)\\ &= 2\sin\omega \,.\, \left(\sin\frac{u}{2}\right)^2\end{aligned}$$

$$\therefore\ y \,.\, \left\{1 + \frac{y}{2} \,.\, \tan\omega\right\} = 2\tan\omega \,.\, \left(\sin\frac{u}{2}\right)^2;$$

$$\therefore\ y = \frac{2.\tan\omega \,.\, \left(\sin\frac{u}{2}\right)^2}{\left(1 + \frac{y}{2} \,.\, \tan\omega\right)} = 2\tan\omega \,.\, \left(\sin\frac{u}{2}\right)^2 . \left\{1 + \frac{y}{2}\tan\omega\right\}^{-1}$$

$$= 2\tan\omega \,.\, \left(\sin\frac{u}{2}\right)^2 . \left\{1 - \frac{y}{2} \,.\, \tan\omega + \ldots\right\}$$

Now neglecting the second and all the succeeding terms of the expansion, as being small when compared with 1, we have for a first approximation, (representing by $y_{\prime}$ this first approximation to the real value of y),

$$y_{\prime} = 2\tan\omega \,.\, \left(\sin\frac{u}{2}\right)^2;$$

And in the second member of the equation putting for y this its *first* approximate value, we have for a *second* approximation,

$$\text{Since } y = 2\tan\omega . \left(\sin\frac{u}{2}\right)^2 . \left\{1 - \frac{y}{2} . \tan\omega\right\}, \text{ nearly,}$$

$$\therefore\ y = 2\tan\omega . \left(\sin\frac{u}{2}\right)^2 . \left\{1 - \frac{y_{,}}{2} . \tan\omega\right\}, \text{ nearly,}$$

$$= 2\tan\omega . \left(\sin\frac{u}{2}\right)^2 . \left\{1 - \tan\omega . \left(\sin\frac{u}{2}\right)^2 . \tan\omega\right\}$$

$$= 2\tan\omega . \left(\sin\frac{u}{2}\right)^2 . \left\{1 - (\tan\omega)^2 . \left(\sin\frac{u}{2}\right)^2\right\}.$$

The number of seconds contained in y, is (Art. 126.)

$$\frac{y}{\sin 1''} = 2 . \left\{\frac{\tan\omega}{\sin 1''} . \left(\sin\frac{u}{2}\right)^2 - \frac{(\tan\omega)^3}{\sin 1''}\left(\sin\frac{u}{2}\right)^4\right\};$$

and the two terms of this expression having been separately determined by means of logarithms, the number of seconds in y is known.

Hymers' Astronomy, Art. 148.

Ex. 2. In the same manner it may be shewn from

$$\text{Cos}\,(x + y) = \sin n . \sin x . \cos m + \cos x . \cos n,$$

when y and n are very small, that

$$y = -n . \frac{\cos m - \frac{n}{2} . \cot x}{1 + \frac{y}{2} . \cot x}$$

$$= -n . \left(\cos m - \frac{n}{2} . \cot x\right) . \left(1 - \frac{y}{2} . \cot x\right), \text{ nearly;}$$

whence, neglecting $\frac{n}{2}.\cot z$ with respect to $\cos m$, and $\frac{y}{2}.\cot z$ with respect to 1, we get for a first approximation,

$$y_{,} = -n.\cos m,$$

And for a second approximation,

$$y = -n\left(\cos m - \frac{n}{2}.\cot z\right).\left(1 + \frac{n}{2}.\cos m.\cot z\right)$$

$$= -n.\cos m + \frac{n^2}{2}\cot z.(\sin m)^2,$$

neglecting powers of n above the second.

The number of seconds in y is

$$\frac{y}{\sin 1''} = -\frac{n.\cos m}{\sin 1''} + \frac{n^2}{2.\sin 1''}.\cot z.(\sin m)^2,$$

the terms of this expression are, as in Ex. 1, determined separately by means of logarithms.

Hymers' Astronomy, Art. 124.

131. *To expand, independently of Demoivre's Theorem,* Sin θ *and* Cos θ *in series ascending by powers of* θ, *assuming that* $\frac{\sin\theta}{\theta}$ *becomes* 1, *when* θ *becomes* 0.

If $\sin\theta$ can be expanded in a series ascending by integral powers of θ, such a series cannot possibly contain any but odd powers of θ. For since $\sin\theta = -\sin(-\theta)$, it appears that when θ is changed into $-\theta$, the sine of θ continues of the same magnitude, but changes its sign,—a result which cannot take place if the series contain either constant terms, or terms involving even powers of the angle.

Let therefore

$$\text{Sin}\,\theta = a_1\,\theta + a_3\,\theta^3 + a_5\,\theta^5 + \ldots.$$

$$\therefore\ \frac{\sin\theta}{\theta} = a_1 + a_3\,\theta^2 + a_5\,\theta^4 + \ldots$$

whatever be the value of θ.

And if θ become 0, this equation becomes

$$1 = a_1\,;$$

$\therefore$ in all cases,

$$\text{Sin}\,\theta = \theta + a_3\,\theta^3 + a_5\,\theta^5 + \ldots\ldots$$

Again, since $\cos\theta = \cos(-\theta)$, it follows that the series representing the value of θ cannot involve odd powers of θ: since if the series *did* involve such powers of θ, it would not continue of the same magnitude and sign, when θ was changed into $-\theta$.

Let $\therefore$ $\text{Cos}\,\theta = a_0 + a_2\,\theta^2 + a_4\,\theta^4 + \ldots.$

And if 0 be written for θ, this equation becomes, $1 = a_0$;

$$\therefore\ \text{Cos}\,\theta = 1 + a_2\,\theta^2 + a_4\,\theta^4 + \ldots\ldots$$

Adding these values of Sin θ and Cos θ, we have

$$1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + \ldots = \cos\theta + \sin\theta\,;$$

writing $\theta + \phi$ for θ,

$$\begin{aligned} &1 + (\theta + \phi) + a_2\,(\theta + \phi)^2 + a_3\,(\theta + \phi)^3 + \ldots \\ &\quad = \cos(\theta + \phi) + \sin(\theta + \phi) \\ &\quad = \cos\phi\,.\,\{\cos\theta + \sin\theta\} + \sin\phi\,.\,\{\cos\theta - \sin\theta\} \\ &\quad = \{1 + a_2\phi^2 + \ldots\}\,.\,\{1 + \theta + a_2\theta^2 + a_3\theta^3 + \ldots\} \\ &\qquad\qquad + \{\phi + a_3\phi^3 + \ldots\}\,.\,\{1 - \theta + a_2\theta^2 - a_3\theta^3 + \ldots\}, \end{aligned}$$

and equating the coefficients of $\phi\theta$, $\phi\theta^2$, $\phi\theta^3$, $\phi\theta^4$, we have

$$2a_2 = -1; \qquad \therefore\ a_2 = -\frac{1}{1\,.\,2}$$

$$3a_3 = a_2; \qquad \therefore\ a_3 = -\frac{1}{1\,.\,2\,.\,3}$$

$$4a_4 = -a_3; \qquad \therefore\ a_4 = +\frac{1}{1\,.\,2\,.\,3\,.\,4}$$

$$5a_5 = a_5; \qquad \therefore\ a_5 = +\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,5},$$

and so on;—the law of the coefficients being manifest.

$$\text{Hence}\quad \text{Sin}\,\theta = \theta - \frac{\theta^3}{1\,.\,2\,.\,3} + \frac{\theta^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \dots\dots\dots\dots \quad (1).$$

$$\text{and}\quad \text{Cos}\,\theta = 1 - \frac{\theta^2}{1\,.\,2} + \frac{\theta^4}{1\,.\,2\,.\,3\,.\,4} - \dots\dots\dots\dots \quad (2).$$

132. The expressions of Arts. 105, 107, 118, have been employed in Arts. 124, 127, &c. to expand certain quantities in the form of a series. The operation can be reversed, as in the following instances.

To find the sum of the series,

(1). $\text{Sin}\,a + \sin 2a + \sin 3a + \dots \qquad + \sin na.$

(2). $\text{Cos}\,a + \cos 2a + \cos 3a + \dots \qquad + \cos na.$

(1). Let $2\sqrt{-1}\sin a = x - \dfrac{1}{x}$;

$$\therefore\ 2\sqrt{-1}\sin 2a = x^2 - \frac{1}{x^2} \qquad \text{Art. 107.}$$

$$2\sqrt{-1}\sin 3x = x^3 - \frac{1}{x^3}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$2\sqrt{-1}\sin na = x^n - \frac{1}{x^n};$$

∴ representing the sum of the proposed series by S,

$$2\sqrt{-1}\,S = x + x^2 + x^3 + \dots + x^n$$

$$- \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \dots - \frac{1}{x^n}$$

$$= x\,.\,\frac{x^n - 1}{x - 1} - \frac{1}{x}\,.\,\frac{\dfrac{1}{x^n} - 1}{\dfrac{1}{x} - 1} = \frac{x^{n+1} + \dfrac{1}{x^n} - x - 1}{x - 1}$$

$$= \frac{\left(x^{n+\frac{1}{2}} + \dfrac{1}{x^{n+\frac{1}{2}}}\right) - \left(x^{\frac{1}{2}} + \dfrac{1}{x^{\frac{1}{2}}}\right)}{x^{\frac{1}{2}} - \dfrac{1}{x^{\frac{1}{2}}}}$$

$$= \frac{2\cos\dfrac{(2n+1)a}{2} - 2\cos\dfrac{a}{2}}{2\sqrt{-1}\,\sin\dfrac{a}{2}};$$

$$\therefore\ S = \frac{1}{2\sin\dfrac{a}{2}}\,.\left\{\cos\frac{a}{2} - \cos\frac{(2n+1)a}{2}\right\} = \frac{\sin\dfrac{na}{2}}{\sin\dfrac{a}{2}}\,.\sin\frac{(n+1)a}{2}\,.$$

(2). Similarly,

$$2(\cos a + \cos 2a + \cos 3a + \dots + \cos na)$$

$$= \frac{\left(x^{\frac{2n+1}{2}} - \dfrac{1}{x^{\frac{2n+1}{2}}}\right) - \left(x^{\frac{1}{2}} - \dfrac{1}{x^{\frac{1}{2}}}\right)}{x^{\frac{1}{2}} - \dfrac{1}{x^{\frac{1}{2}}}}$$

$$= \frac{\sin\dfrac{(2n+1)a}{2} - \sin\dfrac{a}{2}}{\sin\dfrac{a}{2}}\,.$$

$$\therefore \text{ the series} = \frac{\cos\frac{(n+1)a}{2} \cdot \sin\frac{na}{2}}{\sin\frac{a}{2}}.$$

133. The formulæ of Chapters II. and III. may sometimes be employed for the like purpose.

To sum the series,

(1). $\text{Sin}\, a + \sin(a+b) + \sin(a+2b) + \ldots \quad + \sin\{a+(n-1).b\}$,

(2). $\text{Cos}\, a + \cos(a+b) + \cos(a+2b) + \ldots \quad + \cos\{a+(n-1).b\}$.

(1). $\text{Cos}(a - \frac{1}{2}b) - \cos(a + \frac{1}{2}b) = 2\sin\frac{1}{2}b \cdot \sin a$ Art. 51, (4).

so $\cos(a+\frac{1}{2}b) - \cos(a+\frac{3}{2}b) = 2\sin\frac{1}{2}b \cdot \sin(a+b)$

$\cos(a+\frac{3}{2}b) - \cos(a+\frac{5}{2}b) = 2\sin\frac{1}{2}b \cdot \sin(a+2b)$

……………………………………………

$$\cos\left(a+\frac{2n-3}{2}.b\right) - \cos\left(a+\frac{2n-1}{2}.b\right) = 2\sin\tfrac{1}{2}b \cdot \sin\{a+(n-1).b\};$$

∴ by addition,

$$2\sin\tfrac{1}{2}b \cdot \sin a + \sin(a+b) + \sin(a+2b) + \ldots$$

$$+ \sin\{a+(n-1).b\} = \cos(a-\tfrac{1}{2}b) - \cos\left(a+\frac{2n-1}{2}.b\right);$$

$$\therefore \text{ the series (1)} = \frac{\sin\frac{nb}{2}}{\sin\frac{b}{2}} \cdot \sin\left(a+\frac{n-1}{2}.b\right).$$

(2). In like manner we should have, beginning with

$$\text{Sin}(a+\tfrac{1}{2}b) - \sin(a-\tfrac{1}{2}b) = 2\sin\tfrac{1}{2}b \cdot \cos a,$$

and proceeding as in the last case,

$$\text{Cos}\, a + \cos(a + b) + \cos(a + 2b) + \ldots + \cos\{a + (n - 1) . b\}$$

$$= \frac{\sin \frac{nb}{2}}{\sin \frac{b}{2}} . \cos\left(a + \frac{n-1}{2} . b\right) .$$

Cor. By writing *a* for *b* in these results, we should immediately obtain the results of Art. 132.

CHAPTER VII.

ON THE SOLUTION OF EQUATIONS AND THE RESOLUTION OF CERTAIN EXPRESSIONS INTO FACTORS.

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134. *To solve a quadratic equation.*

I. Let $x^2 + px - q = 0$ be the equation.

Then $x = -\left\{\frac{p}{2} \mp \sqrt{\left(\frac{p^2}{4} + q\right)}\right\} = -\sqrt{q}\cdot\left\{\frac{p}{2\sqrt{q}} \mp \sqrt{\left(\frac{p^2}{4q} + 1\right)}\right\}$.

$$\text{Let } \frac{4q}{p^2} = (\tan\theta)^2 \text{ .... } (1).$$

$$\therefore \quad x = -\sqrt{q}\cdot\left\{\frac{1}{\tan\theta} \mp \frac{\sqrt{\{(\tan\theta)^2 + 1\}}}{\tan\theta}\right\}$$

$$= -\sqrt{q}\cdot\frac{1 \mp \sec\theta}{\tan\theta}$$

$$= -\sqrt{q}\cdot\frac{\cos\theta \mp 1}{\sin\theta}.$$

$$\text{Now } \frac{\cos\theta - 1}{\sin\theta} = -\frac{2\cdot\left(\sin\frac{\theta}{2}\right)^2}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}} = -\tan\frac{\theta}{2},$$

$$\frac{\cos\theta + 1}{\sin\theta} = \frac{2\left(\cos\frac{\theta}{2}\right)^2}{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}} = \frac{1}{\tan\frac{\theta}{2}}.$$

If therefore x_1 and x_2 be the two roots of the equation,

$$x_1 = \sqrt{q} \,.\, \tan\frac{\theta}{2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$x_2 = \frac{-\sqrt{q}}{\tan\frac{\theta}{2}}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1), we have, $L \tan\theta = \mathrm{l}_{10} 2 + \frac{1}{2} \,.\, \mathrm{l}_{10} q - \mathrm{l}_{10} p + 10,$
which determines θ;

(2), $$\mathrm{l}_{10} x_1 = \frac{1}{2} \,.\, \mathrm{l}_{10} q + L \tan\frac{\theta}{2} - 10,$$

(3), $$\mathrm{l}_{10}(-x_2) = \frac{1}{2} \,.\, \mathrm{l}_{10} q - L \tan\frac{\theta}{2} + 10.$$

II. If the equation be $x^2 - px - q = 0$, the roots are the same as those of the preceding equation with their signs changed; for the product of the roots with their signs changed $(-q)$ remains the same, and the sum of their roots with their signs changed (the coefficient of the second term) changes its sign, but continues of the same magnitude. (*Wood's Algebra*, Art. 281).

III. Let the equation be $x^2 - px + q = 0$; then

$$x = \frac{p}{2} \,.\, \left\{ 1 \pm \sqrt{\left(1 - \frac{4q}{p^2}\right)} \right\}.$$

If $\frac{4q}{p^2}$ be less than 1, assume $\frac{4q}{p^2} = (\sin\theta)^2 \ldots\ldots\ldots\ldots\ldots (1).$

Then, x_1 and x_2 being the roots of the equation,

$$x_1 = \frac{p}{2} \,.\, (1 + \cos\theta) = p \,.\, \left(\cos\frac{\theta}{2}\right)^2 \ldots\ldots\ldots (2).$$

$$x_2 = \frac{p}{2} \,.\, (1 - \cos\theta) = p \,.\, \left(\sin\frac{\theta}{2}\right)^2 \ldots\ldots\ldots (3).$$

From (1), $L \sin\theta = \mathrm{l}_{10} 2 + \frac{1}{2} \cdot \mathrm{l}_{10} q - \mathrm{l}_{10} p + 10,$

(2) $$\mathrm{l}_{10} x_1 = \mathrm{l}_{10} p + 2\, L \cos\frac{\theta}{2} - 20,$$

(3) $$\mathrm{l}_{10} x_2 = \mathrm{l}_{10} p + 2\, L \sin\frac{\theta}{2} - 20.$$

IV. If the equation be $x^2 + px + q = 0$, the roots are the same as those of the last equation with their signs changed. (*Wood's Algebra*, Art. 281.)

In the last two cases, if $\frac{4q}{p^2}$ be > 1, let $\frac{4q}{p^2} = (\sec\theta)^2$;

then $x = \frac{p}{2} \cdot \left\{1 \pm \sqrt{-\left[(\sec\theta)^2 - 1\right]}\right\} = \frac{p}{2} \cdot \left\{1 \pm \tan\theta\sqrt{-1}\right\}$,

and both roots of the equation are impossible.

These solutions may be employed in preference to the common method of solution when p and q contain a great number of figures.

Ex. *Required the roots of* $x^2 + 365{\cdot}42x - 3469{\cdot}1 = 0$.

By case (I.) we have

$$L \tan\theta = l_{10} 2 + \tfrac{1}{2} \cdot l_{10} 3469{\cdot}1 - l_{10} 365{\cdot}42 + 10.$$

By the Tables,

$$\begin{array}{rr}
l_{10} 2 = & {\cdot}3010300 \\
\tfrac{1}{2} \cdot l_{10} 3469{\cdot}1 = \tfrac{1}{2} \times 3{\cdot}5402168 = & 1{\cdot}7701084 \\
 & 10 \\
\hline
 & 12{\cdot}0711384 \\
\text{Subtract } l_{10} 365{\cdot}42 = & 2{\cdot}5627923 \\
\hline
 & 9{\cdot}5083461 = L \tan 17^{\circ},\ 52',\ \text{nearly;}
\end{array}$$

$$\therefore\ l_{10} x_1 = \tfrac{1}{2} \cdot l_{10} 3469{\cdot}1 + L \tan 8^{\circ},\ 56' - 10.$$

$$\begin{array}{rr}
\tfrac{1}{2} \cdot l_{10} 3469{\cdot}1 = & 1{\cdot}7701084 \\
L \tan 8^{\circ},\ 56' = & 9{\cdot}1964302 \\
\hline
 & 10{\cdot}9665386 \\
\text{Subtract} & 10. \\
\hline
 & {\cdot}9665386 = l_{10} 9{\cdot}2584,\ \text{nearly.}
\end{array}$$

Again, $l_{10}(-x_2) = \frac{1}{2} . l_{10} 3469\cdot 1 - L \tan 8^0, 56' + 10.$

$$10 + \frac{1}{2} . l_{10} 3469\cdot 1 = 11\cdot 7701084$$
$$L \tan 8^0, 56' = \underline{9\cdot 1964302}$$
$$2\cdot 5736782 = l_{10} 374\cdot 69, \text{ nearly.}$$

Hence, the approximate values of the roots of the proposed equation are,

$$9\cdot 2584, \text{ and } -374\cdot 69.$$

We might, after finding x_1, have determined x_2 from the equation

$$-(x_1 + x_2) = 365\cdot 42.$$

Wood's Algebra, Art. 271.

135. *To solve a cubic equation.*

Let the equation when transformed, if necessary, to another which wants the second term (*Wood's Algebra*, Art. 284), be

$$x^3 - qx - r = 0;$$

if $x = \frac{y}{n}$, this equation becomes

$$y^3 - qn^2 y - rn^3 = 0.$$

Now $(\cos\phi)^3 - \frac{3}{4}\cos\phi - \frac{\cos 3\phi}{4} = 0.$ Art. 55.

And these equations are identical, if

(1) $\operatorname{Cos}\phi = y.$

(2) $\frac{3}{4} = qn^2$; and $\therefore n = \frac{1}{2}\sqrt{\frac{3}{q}}$;

(3) $\frac{\operatorname{Cos} 3\phi}{4} = rn^3$; and $\therefore \operatorname{Cos} 3\phi = 4rn^3 = \frac{r}{2}\sqrt{\frac{27}{q^3}}.$

Let a be the least value of 3ϕ which satisfies the equation (3); then one value of y is $\cos\phi$, or $\cos\frac{a}{3}$.

Also since, Art. 104, $\cos a = \cos(2m\pi \pm a)$, the two other values of y are contained among the values of $\cos\frac{2m\pi \pm a}{3}$.

Now m, being an integer, must be of one of the forms $3p$, $3p \pm 1$.

$$\text{and } \cos\frac{2 . 3p . \pi \pm a}{3} = \cos\left(2p\pi \pm \frac{a}{3}\right) = \cos\frac{a}{3}. \quad \text{Art. 104.}$$

$$\cos\frac{2(3p \pm 1)\pi \pm a}{3} = \cos\left(2p\pi \pm \frac{2\pi \pm a}{3}\right) = \cos\frac{2\pi \pm a}{3}.$$

Wherefore the three values of y are

$$\cos\frac{a}{3}; \quad \cos\frac{2\pi + a}{3}; \quad \cos\frac{2\pi - a}{3};$$

and the three values of x, or $\frac{y}{n}$, are

$$2\sqrt{\frac{q}{3}} . \cos\frac{a}{3}; \quad 2\sqrt{\frac{q}{3}} . \cos\frac{2\pi + a}{3}; \quad 2\sqrt{\frac{q}{3}} . \cos\frac{2\pi - a}{3}.$$

136. *The solution of a cubic equation given in the last Article applies to those cases only where all the roots are possible:* i. e. *to the irreducible case of Cardan's rule.*

Wood's Algebra, Art. 331.

Since $\text{Cos}\, a$ is always less than 1;

$$\therefore \frac{r}{2}\sqrt{\frac{27}{q^3}} < 1$$

$$\frac{r}{2} < \sqrt{\frac{q^3}{27}}$$

$$\frac{r^2}{4} < \frac{q^3}{27},$$

which is the irreducible case of Cardan's rule.

137. *To resolve the equation* $x^{2n}-1=0$, *into its quadratic factors.*

$$x^{2n}-1=0;$$

$$\therefore\ x^{2n}=1.$$

Now $(\cos\theta+\sqrt{-1}\,.\sin\theta)^{2n}=\cos 2n\theta+\sqrt{-1}\,.\sin 2n\theta$,

and if $\theta=\dfrac{m\pi}{n}$, m and n being integers,

$$\left(\cos\frac{m\pi}{n}+\sqrt{-1}\,.\sin\frac{m\pi}{m}\right)^{2n}=\cos 2m\pi+\sqrt{-1}\,.\sin 2m\pi$$

$$=1,\quad \text{since } \sin 2m\pi=0;$$

$$=x^{2n};$$

$$\therefore\ x=\cos\frac{m\pi}{n}+\sqrt{-1}\,.\sin\frac{m\pi}{n}.$$

Now m, being an integer, must be of the form $p\,.\,2n+r$,

where p is 0, or any integer,

and r is 0, or any integer less than $2n$.

Wherefore $x=\cos\dfrac{(p\,.\,2n+r)\pi}{n}+\sqrt{-1}\,.\sin\dfrac{(p\,.\,2n+r)\pi}{n}$

$$=\cos\left(2p\pi+\frac{r\pi}{n}\right)+\sqrt{-1}\,.\sin\left(2p\pi+\frac{r\pi}{n}\right)$$

$$=\cos\frac{r\pi}{n}+\sqrt{-1}\,.\sin\frac{r\pi}{n}.\quad \text{Art. 104.}$$

And for r writing $0, 1, 2, \ldots 2n-1$, successively in

$$x = \cos\frac{r\pi}{n} + \sqrt{-1}\sin\frac{r\pi}{n};$$

(1) If $r = 0$, $\quad x = 1$,

(2) $\quad r = 1$, $\quad x = \cos\frac{\pi}{n} + \sqrt{-1}\,.\sin\frac{\pi}{n}$,

(3) $\quad r = 2$, $\quad x = \cos\frac{2\pi}{n} + \sqrt{-1}\,.\sin\frac{2\pi}{n}$

....

(n + 1) $\quad r = n$, $\quad x = -1$,

....

(2n − 1) $\quad r = 2n-2$, $\quad x = \cos\frac{2n-2}{n}\pi + \sqrt{-1}\,.\sin\frac{2n-2}{n}\pi$

$$= \cos\frac{2\pi}{n} - \sqrt{-1}\,.\sin\frac{2\pi}{n}.$$ Art. 104.

(2n) $\quad r = 2n-1$, $\quad x = \cos\frac{2n-1}{n}\pi + \sqrt{-1}\,.\sin\frac{2n-1}{n}\pi$

$$= \cos\frac{\pi}{n} - \sqrt{-1}\,.\sin\frac{\pi}{n}.$$

Now the equations (1) and (n + 1) give the quadratic factor

$$(x-1)\,.\,(x+1) = x^2 - 1;$$

the equations (2) and (2n) give

$$\left(x - \cos\frac{\pi}{n} - \sqrt{-1}\,.\sin\frac{\pi}{n}\right)\left(x - \cos\frac{\pi}{n} + \sqrt{-1}\,.\sin\frac{\pi}{n}\right)$$

$$= x^2 - 2\left(\cos\frac{\pi}{n}\right).\,x + 1,$$

and so on;

wherefore we have

$$x^{2n} - 1 = (x^2-1)\,.\left\{x^2 - 2\left(\cos\frac{\pi}{n}\right).\,x + 1\right\}\ldots\left\{x^2 - 2\left(\cos\frac{n-1}{n}\pi\right).\,x + 1\right\}.$$

138. *To resolve the equation* $x^{2n}+1=0$ *into its quadratic factors.*

$$x^{2n}+1=0;$$

$$\therefore\ x^{2n}=-1.$$

Now $(\cos\theta+\sqrt{-1}\,.\sin\theta)^{2n}=\cos 2n\theta+\sqrt{-1}\,.\sin 2n\theta$,

making $2n\theta=(2m+1)\pi$, and proceeding as in the last Article, we have,

$$x=\cos\frac{2m+1}{2n}\pi+\sqrt{-1}\,.\sin\frac{2m+1}{2n}\pi.$$

Also, assuming $m=p\,.\,2n+r$, where p is 0, or any integer, and r is 0, or any integer less than $2n$, we have, as before,

$$x=\cos\frac{2r+1}{2n}\pi+\sqrt{-1}\,.\sin\frac{2r+1}{2n}\pi.$$

(1) If $r=0$, $\quad x=\cos\frac{\pi}{2n}+\sqrt{-1}\,.\sin\frac{\pi}{2n}$;

(2) $\quad r=1$, $\quad x=\cos\frac{3\pi}{2n}+\sqrt{-1}\,.\sin\frac{3\pi}{2n}$;

....

(n) $\quad r=n-1$, $x=\cos\frac{2n-1}{2n}\pi+\sqrt{-1}\,.\sin\frac{2n-1}{2n}\pi$;

(n+1) $r=n$, $\quad x=\cos\frac{2n+1}{2n}\pi+\sqrt{-1}\,.\sin\frac{2n+1}{2n}\pi$

$$=\cos\left(2\pi-\frac{2n-1}{2n}\pi\right)+\sqrt{-1}\,.\sin\left(2\pi-\frac{2n-1}{2n}\pi\right)$$

$$=\cos\frac{2n-1}{2n}\pi-\sqrt{-1}\,.\sin\frac{2n-1}{2n}\pi;$$

(these two are the middle terms of this series of equations)

(2n−1) $r=2n-2,\ x=\cos\dfrac{4n-3}{2n}\pi+\sqrt{-1}\,.\sin\dfrac{4n-3}{2n}\pi$

$$=\cos\frac{3\pi}{2n}-\sqrt{-1}\,.\sin\frac{3\pi}{2n};$$

(2n) $r=2n-1,\ x=\cos\dfrac{\pi}{2n}-\sqrt{-1}\,.\sin\dfrac{\pi}{2n}.$

And forming quadratic factors out of the pairs of simple factors which are equidistant from the extremities of this series of equations, we have

[Since $\left(x-\cos\dfrac{\pi}{2n}+\sqrt{-1}.\sin\dfrac{\pi}{2n}\right).\left(x-\cos\dfrac{\pi}{2n}-\sqrt{-1}.\sin\dfrac{\pi}{2n}\right)$

$=x^2-2\left(\cos\dfrac{\pi}{2n}\right).x+1$; and so with the others;]

$$x^{2n}+1=\left\{x^2-2\left(\cos\frac{\pi}{2n}\right).x+1\right\}.\left\{x^2-2\left(\cos\frac{3\pi}{2n}\right).x+1\right\}\ldots$$

$$\ldots\left\{x^2-2\left(\cos\frac{2n-1}{2n}\pi\right).x+1\right\}.$$

139. *To resolve the equation* $x^{2n+1}-1=0$ *into its quadratic factors.*

As in the last two Articles,

$$(\cos\theta+\sqrt{-1}\,.\sin\theta)^{2n+1}=\cos(2n+1)\theta+\sqrt{-1}\,.\sin(2n+1)\theta$$

$$=1,\ \left[\text{if } (2n+1)\theta=2m\pi,\text{ or } \theta=\frac{2m\pi}{2n+1},\right]$$

$$=x^{2n+1};$$

$$\therefore\ x=\cos\frac{2m}{2n+1}\pi+\sqrt{-1}\,.\sin\frac{2m}{2n+1}\pi;$$

And if $m = p \,.\, (2n+1) + r$; where p is 0 or any integer, and r is 0 or any integer less than $2n+1$, it may be shewn as before, that

$$x = \cos\frac{2r\pi}{2n+1} + \sqrt{-1}\,.\,\sin\frac{2r\pi}{2n+1}.$$

(1) If $r = 0$, $x = 1$.

(2) $r = 1$, $x = \cos\dfrac{2\pi}{2n+1} + \sqrt{-1}\,.\,\sin\dfrac{2\pi}{2n+1}$.

....

(n + 1) $r = n$, $x = \cos\dfrac{2n\pi}{2n+1} + \sqrt{-1}\,.\,\sin\dfrac{2n\pi}{2n+1}$

(n + 2) $r = n+1$, $x = \cos\dfrac{2n+2}{2n+1}\pi + \sqrt{-1}\,.\,\sin\dfrac{2n+2}{2n+1}\pi$

$$= \cos\left(2\pi - \frac{2n\pi}{2n+1}\right) + \sqrt{-1}\,.\,\sin\left(2\pi - \frac{2n\pi}{2n+1}\right)$$

$$= \cos\frac{2n\pi}{2n+1} - \sqrt{-1}\,.\,\sin\frac{2n\pi}{2n+1}$$

....

(2n + 1) $r = 2n$, $x = \cos\dfrac{2\pi}{2n+1} - \sqrt{-1}\,.\,\sin\dfrac{2\pi}{2n+1}$.

Hence, one root is 1, and a factor of the equation is $x - 1$,—and by forming quadratic factors out of equations (2) and (2n + 1),—(3) and (2n),—... we have

$$x^{2n+1} - 1 = (x-1)\,.\,\left\{x^2 - 2\left(\cos\frac{2\pi}{2n+1}\right).\,x + 1\right\}\ldots$$

$$\ldots\left\{x^2 - 2\left(\cos\frac{2n\pi}{2n+1}\right).\,x + 1\right\}.$$

140. *To resolve the equation* $x^{2n+1}+1=0$ *into its factors.*

$$\text{Here} \quad x^{2n+1}=-1,$$

$$(\cos\theta+\sqrt{-1}\,.\sin\theta)^{2n+1}=\cos(2n+1)\,\theta+\sqrt{-1}\,.\sin(2n+1)\,\theta,$$

$$\text{and let} \quad (2n+1)\,\theta=(2m+1)\,\pi, \quad \text{or } \theta=\frac{2m+1}{2n+1}\pi;$$

$$\therefore \quad (\cos\theta+\sqrt{-1}\,.\sin\theta)^{2n+1}=-1$$

$$=x^{2n+1};$$

$$\therefore \quad x=\cos\frac{2m+1}{2n+1}\pi+\sqrt{-1}\,.\sin\frac{2m+1}{2n+1}\pi.$$

Proceeding as in the last Article by making $m=p\,.\,(2n+1)+r$, we have

$$x=\cos\frac{2r+1}{2n+1}\pi+\sqrt{-1}\,.\sin\frac{2r+1}{2n+1}\pi.$$

(1) If $r=0$, $\quad x=\cos\frac{\pi}{2n+1}+\sqrt{-1}\,.\sin\frac{\pi}{2n+1};$

(2) $r=1$, $\quad x=\cos\frac{3\pi}{2n+1}+\sqrt{-1}\,.\sin\frac{3\pi}{2n+1};$

....

(n + 1) $r=n$, $\quad x=\cos\pi+\sqrt{-1}\,.\sin\pi=-1,$

....

(2n) $r=2n-1$, $\quad x=\cos\frac{3\pi}{2n+1}-\sqrt{-1}\,.\sin\frac{3\pi}{2n+1};$

(2n + 1) $r=2n$, $\quad x=\cos\frac{\pi}{2n+1}-\sqrt{-1}\,.\sin\frac{\pi}{2n+1};$

and we have as before,

$$x^{2n+1}+1=\{x+1\}\,.\,\left\{x^2-2\left(\cos\frac{\pi}{2n+1}\right).\,x+1\right\}\ldots\ldots$$

$$\ldots\ldots\left\{x^2-2\left(\cos\frac{2n-1}{2n+1}\pi\right).\,x+1\right\}.$$

141. *To resolve* Sin θ *and* Cos θ *into factors.*

The values of θ which satisfy the equation Sin $\theta = 0$, are

$$0, \quad \pm\pi, \quad \pm 2\pi, \quad \pm 3\pi, \ldots\ldots \pm n\pi, \ldots$$

n being any integer whatever.

Hence, the series which expresses the value of Sin θ is divisible by

$$\theta, \quad \theta-\pi, \quad \theta+\pi, \quad \theta-2\pi, \quad \theta+2\pi, \ldots\ldots$$

Wood's Algebra, Art. 269.

Let therefore

$$\text{Sin}\,\theta = a.\theta.(\theta-\pi).(\theta+\pi).(\theta-2\pi).(\theta+2\pi)\ldots$$

where a is some constant quantity whose value is to be determined.

$$\therefore \text{Sin}\,\theta = \pm a.\theta.(\pi-\theta).(\pi+\theta).(2\pi-\theta).(2\pi+\theta)\ldots\ldots$$

$$= \pm a.\theta.\pi.\left(1-\frac{\theta}{\pi}\right).\pi.\left(1+\frac{\theta}{\pi}\right).2\pi.\left(1-\frac{\theta}{2\pi}\right).2\pi.\left(1+\frac{\theta}{2\pi}\right)\ldots$$

$$= \pm a.\pi^2.2^2\pi^2.3^2\pi^2.\&\text{c.} \times \theta.\left(1-\frac{\theta^2}{\pi^2}\right).\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots$$

$$\therefore \frac{\sin\theta}{\theta} = \pm a.\pi^2.2^2\pi^2.3^2\pi^2.\&\text{c.} \times \left(1-\frac{\theta^2}{\pi^2}\right).\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots$$

Now if θ become 0, $\dfrac{\sin\theta}{\theta}$ becomes 1. Art. 113.

If therefore θ become 0, our equation becomes

$$1 = \pm a.\pi^2.2^2\pi^2.3^2\pi^2.\&\text{c.}$$

$$\therefore \text{Sin}\,\theta = \theta.\left(1-\frac{\theta^2}{\pi^2}\right).\left(1-\frac{\theta^2}{2^2\pi^2}\right).\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots\ldots \quad (1).$$

Again,—the values of θ which satisfy the equation $\text{Cos}\,\theta = 0$, are

$$\pm\frac{\pi}{2},\quad \pm\frac{3\pi}{2},\quad \pm\frac{5\pi}{2},\ldots \pm\frac{2n+1}{2}\pi,\ldots\ldots$$

n being any integer whatever.

Hence we have, as before,

$$\text{Cos}\,\theta = \pm b\,.\frac{\pi^2}{2^2}\,.\frac{3^2\pi^2}{2^2}\,.\ \&\text{c.}\left(1-\frac{2^2\theta^2}{\pi^2}\right).\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\ldots\ldots$$

Now if $\theta = 0$, this equation becomes

$$1 = \pm b\,.\frac{\pi^2}{2^2}\,.\frac{3^2\pi^2}{2^2}\ldots\ldots$$

$$\therefore\ \text{Cos}\,\theta = \left(1-\frac{2^2\theta^2}{\pi^2}\right).\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right).\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right),\ldots\ldots \quad (2).$$

Cor. If $\theta = \frac{\pi}{2}$, the expression for $\text{Sin}\,\theta$ becomes,

$$1 = \frac{\pi}{2}.\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{4^2}\right).\left(1-\frac{1}{6^2}\right).\left(1-\frac{1}{8^2}\right)\ldots.$$

$$= \frac{\pi}{2}.\frac{2^2-1}{2^2}.\frac{4^2-1}{4^2}.\frac{6^2-1}{6^2}.\frac{8^2-1}{8^2}\ldots.$$

$$= \frac{\pi}{2}.\frac{(2-1).(2+1)}{2^2}.\frac{(4-1).(4+1)}{4^2}.\frac{(6-1).(6+1)}{6^2}\ldots$$

$$\therefore\ \pi = 2\,.\frac{2^2}{1.3}.\frac{4^2}{3.5}.\frac{6^2}{5.7}.\frac{8^2}{7.9}\ldots.$$

Wallis's theorem for the determination of π, in which the successive factors become more and more nearly equal to 1.

APPENDIX I.

ON THE LOGARITHMS OF NUMBERS, AND THE CONSTRUCTION OF THE LOGARITHMIC TABLES OF NUMBERS.

1. Def. If $n = a^x$, x is called the *logarithm* of the *number* n to the *base* a; or, *the logarithm of a number to a given base is that power to which the base must be raised to give the number.*

If a logarithmic formula be generally true, whatever value the base may be of, the logarithms of the quantities involved will be written thus, $\log m$, $\log n$. But if the formula be true only when the logarithms are calculated to some particular base, (as 10, for instance), they will be written thus, $\log_{10} m$, $\log_{10} n$; or thus, $l_{10} m$, $l_{10} n$.

If $n = a^x$, and while a remains the same, successive values be given to n, and the corresponding values of x be registered, the tables so formed are called "*tables of a system of logarithms to the base* a".

It will hereafter be shewn, Art. 10, that a system of logarithms calculated to the base 10 is attended with peculiar advantages, and the tables of logarithms in common use are therefore calculated to that base.

2. Since if $n = a^x$, $x = \mathrm{l}_a n$; therefore, in all cases, $n = a^x = a^{\mathrm{l}_a n}$.

Cor. 1. If $x = 1$, n becomes $= a$; and $\therefore \mathrm{l}_a a = 1$.

If $x = 0$, a^x, or n, becomes 1; and $\therefore \mathrm{l}_a 1 = 0$.

Cor 2. If a be the base of any system, since

$$m = a^{\mathrm{l}_a m}, \text{ and } n = a^{\mathrm{l}_a n},$$

$$\therefore m^{\frac{1}{\mathrm{l}_a m}} = a, \text{ and } n^{\frac{1}{\mathrm{l}_a n}} = a,$$

$$\therefore m^{\frac{1}{\mathrm{l}_a m}} = n^{\frac{1}{\mathrm{l}_a n}}$$

$$\therefore m^{\mathrm{l}_a n} = n^{\mathrm{l}_a m}.$$

A true result, whatever be the value of a. Wherefore,

$$m^{\log n} = n^{\log m}.$$

3. *Required from tables of logarithms calculated to a given base, as* ϵ, *to form tables of logarithms to any other base, as* 10.

$$\text{By Art. 2, } \begin{cases} n = \epsilon^{\mathrm{l}_\epsilon n}, \\ n = 10^{\mathrm{l}_{10} n} \\ 10 = \epsilon^{\mathrm{l}_\epsilon 10}; \end{cases}$$

$$\therefore \epsilon^{\mathrm{l}_\epsilon n} = 10^{\mathrm{l}_{10} n} = (\epsilon^{\mathrm{l}_\epsilon 10})^{\mathrm{l}_{10} n} = \epsilon^{\mathrm{l}_\epsilon 10 \cdot \mathrm{l}_{10} n};$$

and equating the indices of ϵ,

$$\mathrm{l}_\epsilon n = \mathrm{l}_\epsilon 10 \,.\, \mathrm{l}_{10} n;$$

$$\therefore \mathrm{l}_{10} n = \frac{1}{\mathrm{l}_\epsilon 10} \,.\, \mathrm{l}_\epsilon n.$$

Hence, the logarithms of any number n *in two systems calculated to any bases, as* 10 *and* ϵ, *are connected by a constant multiplier,* viz. $\frac{1}{\mathrm{l}_\epsilon 10}$;—and therefore from tables of logarithms calculated to a base ϵ, we may form tables to the base 10.

By writing a for 10 in this proof, we get $\mathrm{l}_a n = \dfrac{1}{\mathrm{l}_\epsilon a} \cdot \mathrm{l}_\epsilon n$, or, the constant multiplier connecting the logarithms of a number in the two systems whose bases are a and ϵ, is $\dfrac{1}{\mathrm{l}_\epsilon a}$.

4. *The logarithm of the product of any number of factors is equal to the sum of the logarithms of the several factors.*

$$\text{For } m \cdot n \cdot r \ldots = a^{\mathrm{l}_a m} \cdot a^{\mathrm{l}_a n} \cdot a^{\mathrm{l}_a r} \ldots$$

$$= a^{(\mathrm{l}_a m + \mathrm{l}_a n + \mathrm{l}_a r + \ldots)}$$

$$\text{But } m \cdot n \cdot r \ldots = a^{(\mathrm{l}_a mnr \ldots)};$$

$$\therefore \mathrm{l}_a(m \cdot n \cdot r \ldots) = \mathrm{l}_a m + \mathrm{l}_a n + \mathrm{l}_a r + \ldots$$

$$\text{or, } \log(m \cdot n \cdot r \ldots) = \log m + \log n + \log r + \ldots$$

Hence, if we find in the tables the number whose logarithm is the sum of the logarithms of the several factors, we obtain the product of those factors.

5. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

$$\text{For } a^{\mathrm{l}_a\left(\frac{m}{n}\right)} = \frac{m}{n} = \frac{a^{\mathrm{l}_a m}}{a^{\mathrm{l}_a n}} = a^{\mathrm{l}_a m - \mathrm{l}_a n};$$

$$\therefore \mathrm{l}_a\left(\frac{m}{n}\right) = \mathrm{l}_a m - \mathrm{l}_a n.$$

$$\text{or } \log\left(\frac{m}{n}\right) = \log m - \log n.$$

Hence, if we find in the tables the number whose logarithm is the logarithm of the dividend minus the logarithm of the divisor, we obtain the quotient.

6. *The logarithm of the* c^{th} *power of any number is equal to* c *times the logarithm of the number,—*c *being either whole or fractional.*

$$\text{For } a^{l_a(m^c)} = m^c = (a^{l_a m})^c = a^{c.l_a m};$$

$$\therefore\ l_a(m^c) = c.l_a m;$$

$$\text{or } \log(m^c) = c.\log m.$$

$$\text{Also } a^{l_a(m)^{\frac{1}{c}}} = m^{\frac{1}{c}} = (a^{l_a m})^{\frac{1}{c}} = a^{\frac{1}{c}.l_a m};$$

$$\therefore\ l_a(m^{\frac{1}{c}}) = \frac{1}{c}.l_a m;$$

$$\text{Or, } \log(m^{\frac{1}{c}}) = \frac{1}{c}.\log m.$$

Hence, the c^{th} power of any given number is that number in the tables whose logarithm is c times the logarithm of the given number; and the c^{th} root of it is that number in the tables whose logarithm is $\left(\frac{1}{c}\right)^{\text{th}}$ of the logarithm of the given number.

7. From the last three articles it appears, that tables of logarithms enable us, particularly where the numbers are large, to perform the operations of multiplication, division, involution, and evolution, with greater ease than by the common arithmetical methods. The easier arithmetical operations of addition and subtraction cannot be performed by logarithms.

8. *In the common, or Briggs', system of logarithms, where the base is* 10, *the logarithms of* $10^n \times N$, *and* $\frac{N}{10^n}$, *may be determined from the logarithm of* N.

$$\begin{aligned} \text{For } l_{10}(10^n \times N) &= l_{10}10^n + l_{10}N, && \text{Art. 4.} \\ &= n.l_{10}10 + l_{10}N, && \text{Art. 6.} \\ &= n + l_{10}N. && \text{Art. 2. Cor. 1.} \\ \text{And } l_{10}\left(\frac{N}{10^n}\right) &= l_{10}N - l_{10}10^n, && \text{Art. 5.} \\ &= l_{10}N - n. \end{aligned}$$

Thus we find in the pages of logarithms printed at the end of this Appendix, (p. 164.),

$$\text{l}_{10}6 = 0{\cdot}7781513\,;$$

$$\begin{aligned}
\therefore\ \text{l}_{10}60 &= \text{l}_{10}10 + \text{l}_{10}6 &&= 1 + 0{\cdot}7781513 = 1{\cdot}7781513,\\
\text{l}_{10}600 &= \text{l}_{10}(10^2) + \text{l}_{10}6 &&= 2 + 0{\cdot}7781513 = 2{\cdot}7781513.\\
\text{l}_{10}{\cdot}6 &= \text{l}_{10}6 - \text{l}_{10}10 &&= 0{\cdot}7781513 - 1,\\
\text{l}_{10}{\cdot}006 &= \text{l}_{10}6 - \text{l}_{10}(10^3) &&= 0{\cdot}7781513 - 3.
\end{aligned}$$

The last two logarithms are thus written, $\bar{1}{\cdot}7781513$, $\bar{3}{\cdot}7781513$.

DEF. The *integral* part of the logarithm is called *the characteristic of the logarithm of the number;* the *decimal* part is called *the mantissa* of the significant digits* of which the number is composed.

Thus, in $\text{l}_{10}600 = 2{\cdot}7781513$, 2 is the *characteristic* of the *logarithm* of the *number* 600, ·7781513 is the *mantissa* of the *significant digit* 6.

9. *In the common system, to determine the characteristic of the logarithm of any given number.*

If a number be between

1 and 10,	its log. is between 0 and 1;	∴ its characteristic is 0.
10 and 100,	.... 1 and 2;	∴ .... 1.
100 and 1000,	.... 2 and 3;	∴ .... 2.
10^{n-1} and 10^n	.... $n-1$ and n;	∴ .... $n-1$.

Hence the characteristic of the logarithm of a number between 10^{n-1} and 10^n, (which has n integral places), is $n-1$, or is less by one than the number of integral places which the number contains.

Again, if the number be between

1 and $\frac{1}{10}$,	its log. is between 0 and -1;	∴ its characteristic is $\bar{1}$,
$\frac{1}{10}$ and $\frac{1}{100}$,	.... -1 and -2;	∴ .... $\bar{2}$,
$\frac{1}{10^{n-1}}$ and $\frac{1}{10^n}$,	.... $-(n-1)$ and $-n$;	∴ .... $\bar{n}$.

---

* "MANTISSA;" a handful thrown in over and above the exact weight; an overplus.

Hence, the characteristic of the logarithm of a decimal fraction having $n - 1$ cyphers after the decimal point, is $\bar{n}$.

Generally therefore, ***the characteristic of the logarithm of any number is the number of its digits minus one,***—where if the number be a decimal fraction, the cyphers which follow the decimal point are alone counted, and are reckoned negatively.

Conversely therefore, if logarithms be given having characteristics $1, 2, 3 \ldots \bar{1}, \bar{2}, \bar{3} \ldots$ there are in the integral parts of the numbers to which these logarithms belong, $2, 3, 4 \ldots\ldots 0, -1, -2 \ldots$ digits respectively.

Thus the logarithms of 254 and 25400 have for characteristics 2 and 4, and the characteristics of the logarithms of 2·54, 25·4, ·000254 are $0, 1, \bar{4}$.

The mantissa given in the tables for $l_{10} 3652$ is ·5625308. (p. 165).

$$\therefore\ l_{10} 3652 = 3 \cdot 5625308, \quad l_{10} 36 \cdot 52 = 1 \cdot 5625308,$$

$$l_{10} 365200 = 5 \cdot 5625308, \quad l_{10} \cdot 3652 = \bar{1} \cdot 5625308,$$

$$l_{10} \cdot 003652 = \bar{3} \cdot 5625308.$$

10. ***The peculiar advantages of Briggs' system of logarithms.*** From the last Article it appears, that if the base of the system be 10, it is requisite to register the ***mantissæ*** only in the tables, because the ***characteristics*** can be determined by counting the digits in the integral part of the number whose logarithm is required. This omission of the characteristics renders the common tables ***less bulky*** than those calculated to any other base.

Also, from Art. 8, it appears, that in this system the mantissa of N is also the mantissa of $10^n \times N$, and of $\frac{N}{10^n}$, where n is any integer:—this circumstance renders the common tables ***more comprehensive*** than if any other base were taken;—for if any other base were used, the mantissa of N would ***not*** be the mantissa of $10^n . N$, or of $\frac{N}{10^n}$.

In the same way it might be shewn that if our arithmetic were duodecimal, instead of being decimal, tables calculated to the base 12 would possess the same advantages which have been here proved to belong to the tables in common use.

11. The tables of logarithms in common use register, either to *five*, or to *seven*, places of decimals, the mantissæ for numbers from 1 to 100000. There are printed at the end of this Appendix two pages of logarithms in which the mantissæ are calculated to *seven* places of decimals. The line at the top of the second page begins with the number 3650, or 36500, and its mantissa ·5622929 is placed opposite to it. And because the mantissæ of all numbers from 36500 to 36559, (comprised in the first six lines of the page), have the same initial three figures, viz. ·562, these three figures are registered, once for all, opposite to the number 3650, and the four last figures of each succeeding mantissa are placed under the number to which they belong. Thus the first line of the page

Num.	0	1	2	3	4	5	6	7	8	9
3650	5622929	3048	3167	3286	3405	3524	3642	3761	3880	3999

gives us, mantissa of 36500 = ·562 2929

.... 36501 = ·562 3048

.... 36502 = ·562 3167

.... 36503 = ·562 3286,

and so on.

In the same manner, the next line gives the mantissæ of numbers from 36510 to 36519 inclusive.

12. Since a change in the value of the three initial figures may not take place at the *beginning* of one of the horizontal lines, whenever a mantissa is to be taken from the tables, we must be sure to get the *right* initial figures. Thus, (see p. 165), the mantissa of 36643 is ·5639910, and the last four figures of the mantissa of 36644 are put down

as 0029. Now if the mantissæ of these two numbers had the *same* initial three figures, the mantissa of 36644 would be less than that of 36643, which we know cannot possibly be the case. A change in the value of the three initial figures does, in point of fact, take place here,—and the mantissa of 36644, (as do those of the numbers immediately following 36644), begins with ·564, and not with ·563. Similar changes of the initial three figures of the mantissa occur at the numbers 36729, 36813, 36898, 36983, and are indicated by printing in a smaller type the fourth figure of the mantissæ of those numbers.

The construction and use of the small tables in the last column of the page will be explained hereafter.

13. EXAMPLES.

(1). *To multiply* 23 *by* 6. Art. 4.

By the tables, p. 164., Mantissa of 23 is ·3617278,

Mantissa of 16 is ·2041200;

$$\therefore\ \mathrm{l}_{10}23 = 1{\cdot}3617278$$

$$\mathrm{l}_{10}16 = 1{\cdot}2041200$$

$$\overline{2{\cdot}5658478}$$

And, p. 165, the significant digits corresponding to the mantissa ·5658478 are 3680;

$$\therefore\ \mathrm{l}_{10}368{\cdot}0 \text{ or } \mathrm{l}_{10}368 = 2{\cdot}5658478;$$

and 368 is the product sought.

(2). *To multiply* ·0172 *by* ·00241.

$$\mathrm{l}_{10}\ {\cdot}0172 = \bar{2}{\cdot}2355284$$

$$\mathrm{l}_{10}{\cdot}00214 = \bar{3}{\cdot}3304138,$$

$$\bar{5}{\cdot}5659422, = \mathrm{l}_{10}{\cdot}000036808.$$

(3). *To divide 3672 by 51000.* Art. 5.

$$\begin{array}{rll} l_{10}3672 = & 3\cdot5649027 & \text{p. } 165. \\ l_{10}51000 = & 4\cdot7075702 & \text{p. } 164. \\ \hline & \bar{2}\cdot8573325 & \end{array}$$

And, p. 164, we find this mantissa to be that corresponding to the significant digits 72;

$\therefore\ \bar{2}\cdot8573325 = l_{10}\cdot072$; and ·072 is the quotient sought.

(4). *To find the values of* $(15\cdot4)^3$, *and* $(650)^{\frac{1}{5}}$. Art. 6.

$$\begin{array}{rl} l_{10}15\cdot4 = 1\cdot1875207 & \text{p. } 164. \\ 3 & \\ \hline 3\cdot5625621 = l_{10}3652\cdot3, & \text{nearly, p. } 165. \end{array}$$

$\therefore$ 3652·3 is the approximate cube of 15·4.

Again, $l_{10}650 = 2\cdot8129134$, p. 164.

$$\therefore\ \frac{1}{5}\cdot l_{10}650 = \cdot5625826 = l_{10}3\cdot6524, \text{ nearly, p. } 165.$$

$\therefore$ 3·6524 is the approximate fifth root of 650.

(5). *To find the values of* $(\cdot085)^4$, *and of* $(\cdot000065)^{\frac{1}{3}}$.

$$\begin{array}{r} \bar{2}\cdot9294189 = l_{10}\cdot085 \\ 4 \\ \hline \bar{5}\cdot7176756 \end{array}$$

and this is the mantissa of 522006, nearly. Therefore ·0000522006 is the number sought, nearly.

Again, $l_{10}\cdot000065 = \bar{5}\cdot8129134 = \bar{6} + 1\cdot8129134$;

$$\therefore\ 3\overline{)\ \bar{5}\cdot8129134}$$

$$\therefore\ \bar{2}\cdot6043044 = l_{10}\cdot040207, \text{ nearly.}$$

14. *To expand* $\mathrm{l}_a(1+x)$ *in a series ascending by powers of* x.

Now $a^0=1$, or $\mathrm{l}_a 1=0$; and $\mathrm{l}_a(1+x)$ becomes $\mathrm{l}_a 1$ when x vanishes,—the series therefore which expresses the value of $\mathrm{l}_a(1+x)$ cannot involve any negative power of x, (or it would become infinite when x vanished,) nor can it have a constant term, (or it would not vanish along with x).

Let then $Ax+Bx^2+Cx^3+\ldots=\mathrm{l}_a(1+x)$; (1).

$\therefore$ also, $A(x+h)+B(x+h)^2+C(x+h)^3+\ldots=\mathrm{l}_a(1+\overline{x+h})$, by writing $x+h$ instead of x.

By subtraction,

$$A.\{(x+h)-x\}+B.\{(x+h)^2-x^2\}+C.\{(x+h)^3-x^3\}+\ldots$$

$$\text{or } Ah+B.(2xh+h^2)+C.(3x^2h+3xh^2+h^3)+\ldots$$

$$=\mathrm{l}_a(1+\overline{x+h})-\mathrm{l}_a(1+x)$$

$$=\mathrm{l}_a\frac{1+\overline{x+h}}{1+x}, \quad \text{by Art. 5.}$$

$$=\mathrm{l}_a\left(1+\frac{h}{1+x}\right)$$

$$=A.\frac{h}{1+x}+B.\left(\frac{h}{1+x}\right)^2+C.\left(\frac{h}{1+x}\right)^3+\ldots \quad \text{by (1).}$$

Dividing both sides of this equation by h, we have,

$$A+B.(2x+h)+C.(3x^2+3xh+h^2)+\ldots$$

$$=\frac{A}{1+x}+B.\frac{h}{(1+x)^2}+\ldots$$

Making $h=0$,

$$A+2Bx+3Cx^2+\ldots=\frac{A}{1+x}$$

$$=A.(1-x+x^2-x^3+\ldots)$$

Equating coefficients of like powers of x,

$$2B = -A, \qquad 3C = A, \qquad 4D = -A, \text{ \&c.}$$

$$\therefore\ B = -\frac{A}{2}, \qquad C = \frac{A}{3}, \qquad D = -\frac{A}{4}, \text{ \&c.}$$

$$\therefore\ \mathrm{l}_a(1+x) = A\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) \ldots\ldots\ldots\ldots\ldots \text{(i.)}$$

Cor. 1. Since A is a constant quantity, entirely dependent on the value of a, let ϵ be the quantity whose value is such that $\epsilon^{\frac{1}{A}} = a$;

$$\therefore\ \frac{1}{A} = \mathrm{l}_\epsilon a, \quad \text{or } A = \frac{1}{\mathrm{l}_\epsilon a}.$$

$$\therefore\ \mathrm{l}_a(1+x) = \frac{1}{\mathrm{l}_\epsilon a}\cdot\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) \ldots\ldots\ldots\ldots \text{(ii.)}$$

and if the base be ϵ,

$$\mathrm{l}_\epsilon(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii.)}$$

We shall hereafter find the value of ϵ, which is a constant quantity 2·7182818....Logarithms to the base ϵ are called *Napierian*, from *Napier*, the inventor of logarithms, who adopted this base because logarithms to it are more easily calculated than those to any other base; as is evident from comparing the series (ii.) and (iii.)

Cor. 2. $\mathrm{l}_\epsilon a = \mathrm{l}_\epsilon\{1 + (a-1)\}$; which by (iii.) is

$$= (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \ldots\ldots\ldots\ldots \text{(iv.)}$$

Cor. 3. $A = \dfrac{1}{\mathrm{l}_\epsilon a} = \dfrac{1}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots}$.

Def. The quantity A, or $\dfrac{1}{\mathrm{l}_\epsilon a}$, by which the logarithms calculated to the base ϵ are multiplied to give the logarithms to the base a, [for $\mathrm{l}_a N = \frac{1}{\mathrm{l}_e a} \cdot \mathrm{l}_e N$, Art. 3.] is called the *modulus* of the system whose base is a.

15. We shall next investigate some rapidly converging formulæ for the calculation of logarithms.

By (iii.), $\mathrm{l}_e \frac{1}{x} = \mathrm{l}_e \left(1 - \frac{x-1}{x}\right) = -\frac{x-1}{x} - \frac{1}{2}\left(\frac{x-1}{x}\right)^2 - \dots$

But $\mathrm{l}_e \frac{1}{x} = \mathrm{l}_e 1 - \mathrm{l}_e x = -\mathrm{l}_e x$;

$$\therefore \quad \mathrm{l}_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \quad \dots\dots\dots\dots\dots\dots \text{(v.)}$$

Again, $\mathrm{l}_e (1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$

$\mathrm{l}_e (1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \dots$

$\therefore$ $\mathrm{l}_e (1+z) - \mathrm{l}_e (1-z)$, or $\mathrm{l}_e \frac{1+z}{1-z} = 2 \,.\, \left\{z + \frac{z^3}{3} + \frac{z^5}{5} + \dots\right\}$.

And if $\frac{1+z}{1-z} = x$, and $\therefore$ $z = \frac{x-1}{x+1}$, this becomes,

$$\mathrm{l}_e x = 2 \,.\, \left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \dots\right\} \quad \dots\dots\dots\dots \text{(vi.)}$$

If x be a little greater than 1, this series converges very rapidly.

Again, $\mathrm{l}_e x = \mathrm{l}_e \left(x^{\frac{1}{m}}\right)^m = m \,.\, \mathrm{l}_e x^{\frac{1}{m}}$

$= m \,.\, \left\{\left(x^{\frac{1}{m}} - 1\right) - \frac{1}{2} \,.\, \left(x^{\frac{1}{m}} - 1\right)^2 + \dots\right\}$, by (iv.)

and, (x being greater than 1), by assuming m of sufficient magnitude, $x^{\frac{1}{m}}$ may be made to differ from 1 by any definite quantity, however small the quantity may be; in which case, the succeeding terms of the series may be neglected as being of inconsiderable magnitude with respect to the first, and we have

$$\mathrm{l}_e x = m\,.\,(x^{\frac{1}{m}} - 1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{vii.})$$

16. *Having given* l_ex, *to find* l_e(x + z), z *being small when compared with* x.

$$\mathrm{l}_e(x+z) = \mathrm{l}_e x\,.\left(1+\frac{z}{x}\right)$$

$$= \mathrm{l}_e x + \mathrm{l}_e\left(1+\frac{z}{x}\right)$$

Expanding $\mathrm{l}_e\left(1+\dfrac{z}{x}\right)$ by (vi.), we have

$$\left[\text{since } \frac{\left(1+\frac{z}{x}\right)-1}{\left(1+\frac{z}{x}\right)+1} = \frac{z}{2x+z}\right],$$

$$\mathrm{l}_e(x+z) = \mathrm{l}_e x + 2\left\{\frac{z}{2x+z} + \frac{1}{3}\,.\left(\frac{z}{2x+z}\right)^3 + \ldots\right\}\ldots\ldots(\text{viii.})$$

Cor. If $z = 1$,

$$\mathrm{l}_e(1+x) = \mathrm{l}_e x + 2\left\{\frac{1}{2x+1} + \frac{1}{3}\,.\,\frac{1}{(2x+1)^3} + \frac{1}{5}\,.\,\frac{1}{(2x+1)^5} + \ldots\right\}\ldots.(\text{ix.})$$

which is useful in computing $\mathrm{l}_e(1+x)$ from $\mathrm{l}_e x$, particularly when x is large.

17. *Having given the Napierian logarithms of two successive numbers,* x − 1 *and* x, *to find that of the number next following.*

$$\mathrm{l}_e(x+1) = \mathrm{l}_e\frac{x^2-1}{x-1}$$

$$= \mathrm{l}_e\frac{x^2.\left(1-\dfrac{1}{x^2}\right)}{x-1}$$

$$= 2\,\mathrm{l}_e x - \mathrm{l}_e(x-1) + \mathrm{l}_e\left(1-\frac{1}{x^2}\right).$$

Expanding $\mathrm{l}_e\left(1-\dfrac{1}{x^2}\right)$ by (vi.) we have,

$$\left[\text{since } \frac{\left(1-\dfrac{1}{x^2}\right)-1}{\left(1-\dfrac{1}{x^2}\right)+1} = -\frac{1}{2x^2-1}\right],$$

$$\mathrm{l}_e(x+1) = 2.\mathrm{l}_e x - \mathrm{l}_e(x-1) - 2.\left\{\frac{1}{2x^2-1} + \tfrac{1}{3}.\frac{1}{(2x^2-1)^3} + \ldots\right\}. \quad \text{(x.)}$$

18. *To expand* a^x *in a series ascending by powers of* x; i. e. *to expand the number in a series ascending by powers of the logarithm.*

$$a^x = \{1 + (a-1)\}^x$$

$$= 1 + x.(a-1) + x.\frac{x-1}{2}.(a-1)^2 + x.\frac{(x-1)}{2}.\frac{(x-2)}{3}.(a-1)^3 + \ldots$$

let the coefficient of x, which is $(a-1) - \tfrac{1}{2}.(a-1)^2 + \tfrac{1}{3}.(a-1)^3 - \ldots$, be represented by p_1, and let the coefficients of x^2, x^3, ... be represented by p_2, p_3, ...;—then we have

$$a^x \quad = 1 + p_1 x + p_2 x^2 + p_3 x^3 + \dots$$

$$\therefore\ a^z \quad = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$$

Now $a^{(x+z)} = a^x . a^z$,

or $1 + p_1 \cdot (x + z) + p_2 \cdot (x + z)^2 + \dots + p_n \cdot (x + z)^n + \dots$

$$= \left\{1 + p_1 x + p_2 x^2 + \dots + p_n x^n + \dots\right\} \times \left\{1 + p_1 z + p_2 z^2 + \dots + p_n z^n + \dots\right\},$$

and equating the coefficients of the terms involving xz, $x^2 z$, $x^3 z$, ... $x^{n-1} z$, ... we have

$$2 p_2 = p_1 \cdot p_1, \qquad \therefore\ p_2 = \frac{(p_1)^2}{2}$$

$$3 p_3 = p_1 \cdot p_2, \qquad \therefore\ p_3 = \frac{(p_1)^3}{1 . 2 . 3}$$

$$4 p_4 = p_1 \cdot p_3, \qquad \therefore\ p_4 = \frac{(p_1)^4}{1 . 2 . 3 . 4}$$

$$\dots\dots\dots\dots \qquad \dots\dots\dots\dots$$

$$n . p_n = p_1 \cdot p_{n-1}, \qquad \therefore\ p_n = \frac{(p_1)^n}{1 . 2 . 3 \dots n}.$$

Now $p_1 = (a - 1) - \frac{1}{2} \cdot (a - 1)^2 + \frac{1}{3} \cdot (a - 1)^3 - \dots$

$$= \text{l}_\epsilon a; \quad \text{by (iv.)}$$

$$\therefore\ a^x = 1 + \frac{\text{l}_\epsilon a}{1} . x + \frac{(\text{l}_\epsilon a)^2}{1 . 2} . x^2 + \frac{(\text{l}_\epsilon a)^3}{1 . 2 . 3} . x^3 + \dots + \frac{(\text{l}_\epsilon a)^n}{1 . 2 . 3 \dots n} . x^n + \dots \text{(xi.)}$$

Cor. If the base be ϵ, (xi) becomes

$$\epsilon^x = 1 + x + \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3} + \dots + \frac{x^n}{1 . 2 \dots n} + \dots\dots \text{(xii.)}$$

19. *To find the value of ε, the base of the Napierian system of logarithms.*

If $x=1$, and the base be ε, the series for a^x in the last Article becomes

$$\epsilon = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots + \frac{1}{1.2.3\ldots n} + \ldots$$

$$\begin{array}{lll}
\text{Now,} & 1+1 & = 2\cdot \\
 & \dfrac{1}{1.2} & = \cdot 5 \\
 & \dfrac{1}{1.2.3} & = \cdot 1666666666 \\
 & \dfrac{1}{1.2.3.4} & = \cdot 0416666666 \\
 & \dfrac{1}{1.2.3.4.5} & = \cdot 0083333333 \\
 & \dfrac{1}{1.2.3.4.5.6} & = \cdot 0013888888 \\
 & \dfrac{1}{1.2.3.4.5.6.7} & = \cdot 0001984126 \\
 & \dfrac{1}{1.2.3.4.5.6.7.8} & = \cdot 0000248015 \\
 & \dfrac{1}{1.2.3\ldots\ldots.8.9} & = \cdot 0000027557 \\
 & \dfrac{1}{1.2.3.4\ldots.9.10} & = \cdot 0000002755 \\
 & & \overline{\ 2\cdot 7182818}
\end{array}$$

This gives the correct value of ε, the base of the Napierian system, so far as the figures are put down. By taking more terms and a greater number of figures in each, we could determine the value of ε to any required degree of accuracy.

20. *On the construction of the common tables of logarithms.*

From one of the series (v), (vi), (vii), the Napierian logarithms of low primes may be found. The logarithm of a high number, which is not a prime, may be determined from the logarithms of its factors, by resolving the number into powers of its prime factors;

Thus, $\mathrm{l}_e 288 = \mathrm{l}_e 2^5 . 3^2 = \mathrm{l}_e 2^5 + \mathrm{l}_e 3^2 = 5 \times \mathrm{l}_e 2 + 2 \times \mathrm{l}_e 3$.

And the expressions (viii), (ix), (x), will greatly facilitate the finding of the logarithms of high numbers which are primes.

The Napierian logarithms having been determined, the tables to base 10 are deduced from them by multiplying each by $\frac{1}{\mathrm{l}_e 10}$, which is equal to ·434294819... Art. 3.

Some of the artifices used in computing the tables may be found in Sharpe's "Method of making Logarithms" prefixed to Sherwin's Tables.

21. In the common tables are registered the mantissæ of the logarithms of numbers of five places of figures, these mantissæ being computed to seven places of decimals; and at the side of each page is placed a "Table of proportional parts," by which, as it will be shewn, may be found the mantissa of the logarithm of a number containing six or seven places of figures: and conversely, if a logarithm be given whose mantissa is not contained exactly in the tables, the number corresponding to it may, by means of these additional tables, be determined to six or seven places of figures.

22. *On the construction and use of the tables of proportional parts.*

Let m_1 and m_2 be the mantissæ of two *consecutive* integral numbers, n and $n+1$, which contain five digits each, and let m be the mantissa of the number $n + \frac{a}{10}$, which contains *six* digits, the last of which (a) is after the decimal point.

Now, since n and $n + \frac{a}{10}$ have the same number of integral places, their logarithms have the same characteristic;

$$\therefore \quad m - m_1 = \mathrm{l}_{10}\left(n + \frac{a}{10}\right) - \mathrm{l}_{10} n$$

$$= \mathrm{l}_{10} \frac{n + \frac{a}{10}}{n}, \text{ Art. 5.} = \mathrm{l}_{10}\left(1 + \frac{a}{10n}\right)$$

which, by expanding $l_{10}\left(1+\frac{a}{10n}\right)$ by (ii.) and neglecting the succeeding terms as being small compared with the first term, becomes

$$=\frac{1}{l_\epsilon 10}\cdot\frac{a}{10n}\text{, nearly.}$$

$$\text{Similarly, } m_2 - m_1 = l_{10}(n+1) - l_{10}n = \frac{1}{l_\epsilon 10}\cdot\frac{1}{n};$$

$$\therefore\quad m - m_1 = (m_2 - m_1)\cdot\frac{a}{10}.$$

Whence m may be found, if m_1, m_2, a, be given,—or a may be determined, when m_1, m_2, m, are known.

23. By the Tables, page 165,

$$\begin{aligned} \text{Mantissa of } 36633 &= m_2 = \cdot 5638725 \\ \ldots\ldots\ldots\ldots\ 36632 &= m_1 = \cdot 5638606 \\ \hline \therefore\quad m_2 - m_1 &= \cdot 0000119 \end{aligned}$$

And in the expression $m - m_1 = (m_2 - m_1)\frac{a}{10}$, writing for a the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 successively, we have

a	$m_2 - m_1$	
1	$\cdot 0000119 \times \frac{1}{10}$ or ·00000119	·0000012 nearly
2	·00000238	·0000024 ......
3	·00000357	·0000036 ......
4	·00000476	·0000048 ......
5	·00000595	·0000060 ......
6	·00000714	·0000071 ......
7	·00000833	·0000083 ......
8	·00000952	·0000095 ......
9	·00001071	·0000107 ......

119	
1	12
2	24
3	36
4	48
5	60
6	71
7	83
8	95
9	107

Now, the "difference" put down in the Tables for numbers near 36600, is 119, and the Table of proportional parts is as in the margin of the larger Table.

We see then, that *the significant digits only* of the whole difference, and of the differences corresponding to the several digits, are inserted in the table of proportional parts. Hence, the following method of constructing tables of proportional parts is evident:

Of the *significant* part of the whole difference point off the last digit as a decimal—this corresponds to the division of the whole difference by 10, when the whole difference is treated as an integer. Multiply this number by 1, 2, 3,...9 successively, and the *whole* numbers thus obtained (the last digit in the integral part being increased by unity where the decimal part is not less than ·5) are the significant parts of the differences for the digits respectively.

Thus, let the significant part of the whole difference be 156.

$15{\cdot}6 \times 1 = 15{\cdot}6 = 16$ nearly.
$\ldots \times 2 = 31.2 = 31$
$\ldots \times 3 = 46.8 = 47$
$\ldots \times 4 = 62.4 = 62$
$\ldots \times 5 = 78{\cdot}0 = 78$

$15{\cdot}6 \times 6 = 93{\cdot}6 = 94$ nearly.
$\ldots \times 7 = 109{\cdot}2 = 109$
$\ldots \times 8 = 124{\cdot}8 = 125$
$\ldots \times 9 = 140{\cdot}4 = 140.$

By constructing a table of this kind we see, that if the digit be given, the difference is immediately known, or vice versâ. To avoid the necessity of performing the operation of subtraction in any particular case, in order to find the whole difference, there is a line in the tables marked at the top with "Diff.", in which the difference is placed opposite to that logarithm at which such difference begins. To know what the difference therefore is in any particular case, it is merely requisite to take the number in this line next above the logarithm in question.

Ex. 1. *To find the number whose logarithm is* $3{\cdot}5677766$.

By the tables, p. 165, the mantissa next below the given mantissa is that of $l_{10}36963$, and the whole difference put down is 117.

$$\text{Mantissa of the given logarithm} = m = {\cdot}5677766$$

$$\text{Mantissa of } l_{10}\,36963 \qquad = m_1 = {\cdot}5677672$$

$$\therefore\ m - m_1 = \qquad 94$$

By the table of proportional parts to "Diff." 117, the difference 94 corresponds to the digit 8,—therefore the significant part of the number sought is 369638; also since the given logarithm has 3 for its characteristic, the number required is $3696{\cdot}38$.

Ex. 2. *Required the logarithm of* $367{\cdot}654$.

By p. 165., $l_{10}367{\cdot}65 = 2{\cdot}5654346$

And "Diff." being 118, Part for $4 = \quad 47$

$\therefore\ l_{10}367{\cdot}654 = 2.5654393$

24. *To find the mantissa of the logarithm of a number which has seven places of digits.*

Let m_1 and m_2 be the mantissæ of n and $n+1$, two successive integers of five places each: let M be that of $n + \frac{a}{10} + \frac{b}{100}$, which has the same number of integral places as n and $n+1$, and also one digit (a) in the place of the tenths, and another (b) in that of the hundredths.

Now, since n and $n + \frac{a}{10} + \frac{b}{100}$ have the same number of integral places, their logarithms have the same characteristic;

$$\therefore\ M - m_1 = \mathrm{l}_{10}\left(n + \frac{a}{10} + \frac{b}{100}\right) - \mathrm{l}_{10} n = \mathrm{l}_{10} \frac{n + \frac{a}{10} + \frac{b}{100}}{n}$$

$$= \mathrm{l}_{10}\left\{1 + \frac{1}{n}\cdot\left(\frac{a}{10} + \frac{b}{100}\right)\right\}$$

$$= \frac{1}{\mathrm{l}_\epsilon 10}\cdot\frac{1}{n}\cdot\left(\frac{a}{10} + \frac{b}{100}\right) \text{ by (ii.),}$$

neglecting the terms of the series after the first.

Similarly, $m_2 - m_1 = \mathrm{l}_{10}(n + 1) - \mathrm{l}_{10} n$

$$= \mathrm{l}_{10}\left(1 + \frac{1}{n}\right) = \frac{1}{\mathrm{l}_\epsilon 10}\cdot\frac{1}{n};$$

$$\therefore\ M - m_1 = (m_2 - m_1)\cdot\left(\frac{a}{10} + \frac{b}{100}\right)$$

$$= (m_2 - m_1)\cdot\frac{a}{10} + \frac{1}{10}\cdot\left\{(m_2 - m_1)\cdot\frac{b}{10}\right\}.$$

Now the first part of the expression, $(m_2 - m_1)\cdot\frac{a}{10}$, is the quantity to be added to m_1 for the *first* additional digit a, Art. 22.; and $(m_2 - m_1)\cdot\frac{b}{10}$ is what *would* have been added, had b been the *first* additional digit, instead of being the *second*:—wherefore the quantity to be added for b, when it is the *second* additional digit, namely,

$$\frac{1}{10}\cdot\left\{(m_2 - m_1)\cdot\frac{b}{10}\right\},$$

is the tenth part of what would have been added, had b been the *first*, instead of being the *second*, additional figure.

The following Examples will explain what has been said.

Ex. 1. *To find* $l_{10}\,3684{\cdot}286$.

By the tables, p. 165, $l_{10}\,3684{\cdot}2 = 3{\cdot}5663432$, and "Diff." being 118, the part for a *first* additional digit 8 is 94; and for a *first* additional digit 6 the part is 71, and therefore the part when 6 is a *second* additional digit is $\frac{71}{10}$, or 7·1. The operation is thus performed;

$$\begin{array}{lrr} & l_{10}\,3684{\cdot}2 = & 3{\cdot}5663432 \\ \text{Part for} & 8 = & 94 \\ \text{Part for} & 6 = & 7{\cdot}1 \\ \hline \therefore & l_{10}\,3684{\cdot}286 = & 3{\cdot}5663533 \end{array}$$

Ex. 2. *To find the number whose logarithm is* 2·5656560.

$$\begin{array}{ll} & 2{\cdot}5656560 \\ \text{Now} & \underline{2{\cdot}5656471} = l_{10}\,367{\cdot}83, \quad \text{p. } 165. \\ & \qquad 89 \\ & \qquad \underline{83} = \text{Part for digit } 7. \quad \text{"Diff." being } 118. \\ & \qquad\ \ 6 = \text{Part for a second digit } 5. \end{array}$$

$\therefore\ 2{\cdot}5656560 = l_{10}\,367{\cdot}8375$,

and therefore 367·8375 is the number sought.

25. *On the adaptation of formulæ to logarithmic computation.*

After a table of logarithms has once been constructed, the labour of going through long arithmetical operations can be materially diminished, while at the same time the chance of committing errors is lessened. But, by referring to Articles 4, 5, 6 of this Appendix, it will appear that the sole arithmetical operations which can be performed by logarithms, are those of

Multiplication, Division, Involution, and Evolution. To avail ourselves therefore of logarithms in calculating the value of an expression, the expression must be put into such a form that no other arithmetic operations than these shall require to be performed. Such an arrangement of an expression is called the *adaptation* of it *to logarithmic computation.*

Thus if a, b, c, the three sides of a triangle, be given, logarithms cannot be *directly* applied to determine the value of $\cos A$ from the equation,

$$\operatorname{Cos} A = \frac{b^2 + c^2 - a^2}{2bc}; \qquad \text{Art. 74, p. 59.}$$

but if from this equation we deduce the formula,

$$\operatorname{Cos} \frac{A}{2} = \sqrt{\frac{S.(S-a)}{bc}}, \text{ where } S = \tfrac{1}{2}(a+b+c), \text{ Art. 75, p. 60.}$$

we can *immediately* apply logarithms to determine the value of $\operatorname{Cos} \frac{A}{2}$.

For a, b, c being given, S and $S-a$ are easily determined,—and these being known, $\cos \frac{A}{2}$ is determined from the equation

$$\begin{aligned} l_{10} \operatorname{Cos} \frac{A}{2} &= \tfrac{1}{2} \{ l_{10} [S.(S-a)] - l_{10} bc \} \\ &= \tfrac{1}{2} \{ l_{10} S + l_{10} (S-a) - l_{10} b - l_{10} c \}. \end{aligned}$$

The two accompanying pages of logarithms are taken from Babbage's Tables, the most correct and the best arranged of any which have been published. The columns are omitted which in those Tables are given to determine the number of seconds in an angle containing a given number of degrees, minutes, and seconds, and conversely.

1	0000000	51	7075702	101	0043214	151	1789769	201	3031961
2	3010300	52	7160033	102	0086002	152	1818436	202	3053514
3	4771213	53	7242759	103	0128372	153	1846914	203	3074960
4	6020600	54	7323938	104	0170333	154	1875207	204	3096302
5	6989700	55	7403627	105	0211893	155	1903317	205	3117539
6	7781513	56	7481880	106	0253059	156	1931246	206	3138672
7	8450980	57	7558749	107	0293838	157	1958997	207	3159703
8	9030900	58	7634280	108	0334238	158	1986571	208	3180633
9	9542425	59	7708520	109	0374265	159	2013971	209	3201463
10	0000000	60	7781513	110	0413927	160	2041200	210	3222193
11	0413927	61	7853298	111	0453230	161	2068259	211	3242825
12	0791812	62	7923917	112	0492180	162	2095150	212	3263359
13	1139434	63	7993405	113	0530784	163	2121876	213	3283796
14	1461280	64	8061800	114	0569049	164	2148438	214	3304138
15	1760913	65	8129134	115	0606978	165	2174839	215	3324385
16	2041200	66	8195439	116	0644580	166	2201081	216	3344538
17	2304489	67	8260748	117	0681859	167	2227165	217	3364597
18	2552725	68	8325089	118	0718820	168	2253093	218	3384565
19	2787536	69	8388491	119	0755470	169	2278867	219	3404441
20	3010300	70	8450980	120	0791812	170	2304489	220	3424227
21	3222193	71	8512583	121	0827854	171	2329961	221	3443923
22	3424227	72	8573325	122	0863598	172	2355284	222	3463530
23	3617278	73	8633229	123	0899051	173	2380461	223	3483049
24	3802112	74	8692317	124	0934217	174	2405492	224	3502480
25	3979400	75	8750613	125	0969100	175	2430380	225	3521825
26	4149733	76	8808136	126	1003705	176	2455127	226	3541084
27	4313638	77	8864907	127	1038037	177	2479733	227	3560259
28	4471580	78	8920946	128	1072100	178	2504200	228	3579348
29	4623980	79	8976271	129	1105897	179	2528530	229	3598355
30	4771213	80	9030900	130	1139434	180	2552725	230	3617278
31	4913617	81	9084850	131	1172713	181	2576786	231	3636120
32	5051500	82	9138139	132	1205739	182	2600714	232	3654880
33	5185139	83	9190781	133	1238516	183	2624511	233	3673559
34	5314789	84	9242793	134	1271048	184	2648178	234	3692159
35	5440680	85	9294189	135	1303338	185	2671717	235	3710679
36	5563025	86	9344985	136	1335389	186	2695129	236	3729120
37	5682017	87	9395193	137	1367206	187	2718416	237	3747483
38	5797836	88	9444827	138	1398791	188	2741578	238	3765770
39	5910646	89	9493900	139	1430148	189	2764618	239	3783979
40	6020600	90	9542425	140	1461280	190	2787536	240	3802112
41	6127839	91	9590414	141	1492191	191	2810334	241	3820170
42	6232493	92	9637878	142	1522883	192	2833012	242	3838154
43	6334685	93	9684829	143	1553360	193	2855573	243	3856063
44	6434527	94	9731279	144	1583625	194	2878017	244	3873898
45	6532125	95	9777236	145	1613680	195	2900346	245	3891661
46	6627578	96	9822712	146	1643529	196	2922561	246	3909351
47	6720979	97	9867717	147	1673173	197	2944662	247	3926970
48	6812412	98	9912261	148	1702617	198	2966652	248	3944517
49	6901961	99	9956352	149	1731863	199	2988531	249	3961993
50	6989700	100	0000000	150	1760913	200	3010300	250	3979400

Num.	0	1	2	3	4	5	6	7	8	9	Diff.	
3650	5622929	3048	3167	3286	3405	3524	3642	3761	3880	3999	119	119 1 12 2 24 3 36 4 48 5 60 6 71 7 83 8 95 9 107
1	4118	4237	4356	4475	4594	4713	4832	4951	5070	5189		
2	5308	5427	5546	5664	5783	5902	6021	6140	6259	6378		
3	6497	6616	6734	6853	6972	7091	7210	7329	7448	7567		
4	7685	7804	7923	8042	8161	8280	8398	8517	8636	8755		
5	8874	8993	9111	9230	9349	9468	9587	9705	9824	9943		
6	5630062	0181	0299	0418	0537	0656	0775	0893	1012	1131		
7	1250	1368	1487	1606	1725	1843	1962	2081	2200	2318		
8	2437	2556	2674	2793	2912	3031	3149	3268	3387	3505		
9	3624	3743	3861	3980	4099	4218	4336	4455	4574	4692		
3660	4811	4930	5048	5167	5285	5404	5523	5641	5760	5879		
1	5997	6116	6235	6353	6472	6590	6709	6828	6946	7065		
2	7183	7302	7421	7539	7658	7776	7895	8013	8132	8251		
3	8369	8488	8606	8725	8843	8962	9081	9199	9318	9436		
4	9555	9673	9792	9910	0029	0147	0266	0384	0503	0621		
5	5640740	0858	0977	1095	1214	1332	1451	1569	1688	1806	118	118 1 12 2 24 3 35 4 47 5 59 6 71 7 83 8 94 9 106
6	1925	2043	2162	2280	2398	2517	2635	2754	2872	2991		
7	3109	3228	3346	3464	3583	3701	3820	3938	4056	4175		
8	4293	4412	4530	4648	4767	4885	5004	5122	5240	5359		
9	5477	5595	5714	5832	5951	6069	6187	6306	6424	6542		
3670	6661	6779	6897	7016	7134	7252	7371	7489	7607	7726		
1	7844	7962	8080	8199	8317	8435	8554	8672	8790	8908		
2	9027	9145	9263	9382	9500	9618	9736	9855	9973	0091		
3	5650209	0328	0446	0564	0682	0800	0919	1037	1155	1273		
4	1392	1510	1628	1746	1864	1983	2101	2219	2337	2455		
5	2573	2692	2810	2928	3046	3164	3282	3401	3519	3637		
6	3755	3873	3991	4109	4228	4346	4464	4582	4700	4818		
7	4936	5054	5173	5291	5409	5527	5645	5763	5881	5999		
8	6117	6235	6353	6471	6590	6708	6826	6944	7062	7180		
9	7298	7416	7534	7652	7770	7888	8006	8124	8242	8360		
3680	8478	8596	8714	8832	8950	9068	9186	9304	9422	9540		
1	9658	9776	9894	0012	0130	0248	0366	0484	0602	0720		
2	5660838	0956	1074	1192	1310	1428	1545	1663	1781	1899		
3	2017	2135	2253	2371	2489	2607	2725	2843	2960	3078		
4	3196	3314	3432	3550	3668	3786	3903	4021	4139	4257		
5	4375	4493	4611	4728	4846	4964	5082	5200	5318	5435		
6	5553	5671	5789	5907	6025	6142	6260	6378	6496	6614		
7	6731	6849	6967	7085	7203	7320	7438	7556	7674	7791		
8	7909	8027	8145	8262	8380	8498	8616	8733	8851	8969		
9	9087	9204	9322	9440	9557	9675	9793	9911	0028	0146		
3690	5670264	0381	0499	0617	0734	0852	0970	1087	1205	1323		
1	1440	1558	1676	1793	1911	2029	2146	2264	2382	2499		
2	2617	2735	2852	2970	3087	3205	3323	3440	3558	3675		
3	3793	3911	4028	4146	4263	4381	4499	4616	4734	4851		
4	4969	5086	5204	5322	5439	5557	5674	5792	5909	6027	117	117 1 12 2 23 3 35 4 47 5 59 6 70 7 82 8 94 9 105
5	6144	6262	6379	6497	6615	6732	6850	6967	7085	7202		
6	7320	7437	7555	7672	7790	7907	8025	8142	8260	8377		
7	8495	8612	8729	8847	8964	9082	9199	9317	9434	9552		
8	9669	9787	9904	0021	0139	0256	0374	0491	0608	0726		
9	5680843	0961	1078	1196	1313	1430	1548	1665	1782	1900		
	0	1	2	3	4	5	6	7	8	9		

APPENDIX II.

ON THE CONSTRUCTION AND USE OF TABLES OF GONIOMETRIC FUNCTIONS.

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1. If it be required to find the value of a trigonometrical formula in which the sines, cosines, tangents, secants, &c. of given angles enter, much labour may evidently be avoided if the values of these quantities be determined, once for all, and registered in tables. Were it necessary on every new occasion that occurred to calculate these quantities afresh, the value of the most simple expression could not be determined without much loss of time and risk of error.

Now in very small angles the sines and tangents are exceedingly small fractions, and if these quantities be expressed as decimal fractions, two or three cyphers will follow the decimal point before we come to the significant digits. Thus the sine of $\frac{30^0}{2^{11}}$ will be proved in the next Article to be ·00025566..., which has *three* cyphers after the decimal point. In order to avoid the inconvenience of printing these cyphers, the real values of all goniometrical quantities are multiplied by 10000, or the decimal point is moved four places to the right, before they are registered in the tables; and the tables so formed are *called*, "Tables of natural sines, cosines, &c.*"

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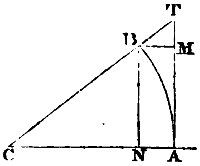
* The tables of goniometric functions are sometimes said to be "calculated to a radius of 10000;" the meaning of this expression we will explain.

If with C as centre, and radius CA, an arc AB be described, and BN, AT be $\perp$ to CA, the *lines* NB, CN, AT, CT, AN respectively, are sometimes defined to be the *sine, cosine, tangent, secant,* and *versed sine, of the angle* ACB *to the radius* CA,—or the *sine, cosine, tangent, secant,* and *versed sine, of the arc* AB



The expression "to the radius CA" is absolutely necessary to the definitions, because the lines NB, AT...evidently depend on the magnitude of AC as well as

2. *To find the values of the sines and cosines of every angle less than* 90^0.

By Art. 39. (2), $2(\cos A)^2 = 1 + \cos 2A$,

$$\text{or } \operatorname{Cos} A = \sqrt{\tfrac{1}{2}(1 + \cos 2A)};$$

$$\therefore \operatorname{Cos}\frac{A}{2} = \sqrt{\tfrac{1}{2}(1 + \cos A)},$$

$$\cos\frac{A}{2^2} = \sqrt{\tfrac{1}{2}\left(1 + \cos\frac{A}{2}\right)}; \text{ and so on.}$$

Now if $A = 30^0$, $\operatorname{Cos} 2A = \frac{1}{2}$. Art. 32. (2). And having performed the successive operations indicated above eleven times, we have

$$\operatorname{Cos}\frac{30^0}{2^{11}}, \quad \text{or } \operatorname{Cos} 52''{\cdot}784375, = {\cdot}99999996782;$$

$$\therefore \operatorname{Sin}\frac{30^0}{2^{11}} = \sqrt{\left\{1 - \left(\cos\frac{30^0}{2^{11}}\right)^2\right\}} = {\cdot}000255663462.$$

3. Now since $\dfrac{30^0}{2^{11}} = \dfrac{30\,.\,60'}{2^{11}}$, and the sines of small angles are as the number of seconds, and therefore as the number of minutes, they contain, Art. 116. Cor.;

upon the magnitude of the angle ACB;—in fact, if the angle ACB continue of the same magnitude, they vary directly as CA.

Now by the definition that we have used of the sine, $\sin ACB = \dfrac{NB}{CB} = \dfrac{NB}{CA}$;

$$\therefore 10000 \times \sin ACB = BN\,.\frac{10000}{CA}.$$

But $10000 \times \sin ACB$ is the tabular sine of ACB;

$$\therefore \text{tab. } \sin ACB = NB\,.\frac{10000}{CA} = NB, \text{ if } CA = 10000.$$

Wherefore the tab.sine of ACB expresses the magnitude of the line NB, (the sine of ACB to the radius CA), the magnitude of CA being represented by 10000.

Similarly the tab.cosine, tab.tangent,...of ACB, express the magnitudes of the lines CN, AT,...the magnitude of CA being 10000.

$$\therefore\ \sin 1' : \sin\frac{30^0}{2^{11}} :: 1' : \frac{30.60'}{2^{11}};$$

$$\therefore\ \text{Sin}\,1' = \frac{2^{11}}{30.60}\sin\frac{30^0}{2^{11}}$$

$$= \frac{2^{8}}{15.15}\sin\frac{30^0}{2^{11}}$$

$$= \frac{256}{225}\times \cdot000255663462$$

$$= \cdot000290888204.$$

And from $\sin 1'$ we may thus determine $\sin 30''$;

$\text{Sin}\,30'' = \sin\frac{1}{2}' = \frac{1}{2}\left\{\sqrt{(1+\sin 1')} - \sqrt{(1-\sin 1')}\right\}$, Art. 41.

4. Sin $1'$ *and* Sin $30''$ *being known, the sines of* $1'$, $2'$, $3'$... *to* $60'$ (*or* 1^0) *may be determined.*

For $\text{Sin}(A+B) + \sin(A-B) = 2\sin A\,.\cos B$

$$= 2\sin A\,.\left\{1 - 2\left(\sin\frac{B}{2}\right)^2\right\}$$

$$= \sin A + \sin A - \sin A\,.\left(2\sin\frac{B}{2}\right)^2,$$

$$\therefore\ \text{Sin}(A+B) = \sin A + \left\{\sin A - \sin(A-B)\right\} - \sin A\,.\left(2\sin\frac{B}{2}\right)^2.$$

If in this formula $B = 1'$, and we write $1'$, $2'$, $3'$...... successively for A, we have

$\text{Sin}\,2' = \sin 1' + (\sin 1' - \sin 0') - \sin 1'\,.(2\sin 30'')^2$,

$\sin 3' = \sin 2' + (\sin 2' - \sin 1') - \sin 2'\,.(2\sin 30'')^2$,

$\sin 4' = \sin 3' + (\sin 3' - \sin 2') - \sin 3'\,.(2\sin 30'')^2$, &c.

This method is not very laborious:—in the last example, $\sin 3'$ and $\sin 3' - \sin 2'$ are known from the two preceding lines; and the only labour is in multiplying $\sin 3'$ by the factor $(2\sin 30'')^2$ which is common to all the terms.

Cor. If in the above formula $B = 1^0$ and we write 1^0, 2^0, 3^0......successively for A, we can determine the sines of 2^0, 3^0, 4^0......

5. *The sines of angles up to* 60^0 *having been calculated, those of angles between* 60^0 *and* 90^0 *may be thus determined.*

$$\text{Sin}\,(60^0 + A) - \sin\,(60^0 - A) = 2\cos 60^0 . \sin A.$$

$$\text{And } \cos 60^0 = \tfrac{1}{2};$$

$$\therefore \text{Sin}\,(60^0 + A) = \sin A + \sin\,(60^0 - A).$$

And if A increase by $1'$ at a time from 0^0 to 30^0, this formula will by addition merely give the sines of angles from 60^0 to 90^0.

6. *The sines of angles up to* 90^0 *having been determined, their cosines are also known.*

$$\text{For } \quad \text{Cos}\,A = \sin\,(90^0 - A).$$

$$\text{Thus } \quad \text{Cos}\,25^0 = \sin\,(90^0 - 25^0) = \sin 65^0,$$

$$\text{Cos}\,72^0 = \sin\,(90^0 - 72^0) = \sin 18^0.$$

7. *The tangents, cotangents, secants, and cosecants can be determined from the sines and cosines.*

$$\text{For } \quad \text{Tan}\,A = \frac{\sin A}{\cos A}, \quad \text{Cot}\,A = \frac{\cos A}{\sin A},$$

$$\text{Sec}\,A = \frac{1}{\cos A}, \quad \text{Cosec}\,A = \frac{1}{\sin A}.$$

8. Since,

$$\text{Sin}\,A = \cos\,(90^0 - A),$$

$$\text{Tan}\,A = \cot\,(90^0 - A),$$

$$\text{Sec}\,A = \text{cosec}\,(90^0 - A),$$

the sines, tangents, and secants of angles from 45^0 to 90^0 are the same respectively as the cosines, cotangents, and cosecants of angles from 45^0 to 0^0. Wherefore it is unnecessary to carry the tables further than to the angle 45^0.

Thus $\text{Cos } 72^{0}, 20' = \sin(90 - 72^{0}, 20') = \sin 17^{0}, 40'$,

$\text{Sin } 72^{0}, 20' = \cos 17^{0}, 40'$,

$\text{Tan } 72^{0}, 20' = \cot 17^{0}, 40'$,

$\text{Sec } 72^{0}, 20' = \operatorname{cosec} 17^{0}, 40'$.

At the bottom of the page containing the functions of angles from 17° to 18° is placed the angle 72°, and the column which at the top of the page is marked to indicate the sines of angles from 17° to 18°, is marked at the bottom to shew the cosines of angles from 72° to 73°; and so for the other functions. See page 173.

9. *On Formulæ of Verification.*

Since the trigonometrical functions of angles are determined *successively* from one another, one error will affect every successive result. As checks against the possibility of errors, several formulæ (*of verification* as they are called) are used to examine the accuracy of the results, and the values registered in the tables are presumed to be correct if they satisfy these formulæ.

The following are the principal formulæ of verification.

(1). $\text{Sin } A = \frac{1}{2}\left\{\sqrt{1 + \sin 2A} - \sqrt{1 - \sin 2A}\right\}$. Art. 41.

(2). $\text{Cos } A = \frac{1}{2}\left\{\sqrt{1 + \sin 2A} + \sqrt{1 - \sin 2A}\right\}$.

A being an angle less than 45°.

Again, $\text{Cos } 36^{0} = \dfrac{\sqrt{5}+1}{4}$; $\text{Cos } 72^{0} = \dfrac{\sqrt{5}-1}{4}$; Art. 58. (3). (1).

$$\left.\begin{aligned} \therefore\ \text{Sin}(36^{0}+A)-\sin(36^{0}-A) &= 2\cos 36^{0}.\sin A = \frac{\sqrt{5}+1}{2}.\sin A, \\ \text{and } \text{Sin}(72^{0}+A)-\sin(72^{0}-A) &= 2\cos 72^{0}.\sin A = \frac{\sqrt{5}-1}{2}.\sin A; \end{aligned}\right\}$$

Art. 51. (2).

$\therefore$ by subtraction we have,

$$\text{Sin}\,(36^\circ + A) + \sin\,(72^\circ - A) - \sin\,(36^\circ - A) - \sin\,(72^\circ + A) = \sin A\,;$$

(3). $\therefore$ $\text{Sin}\,A + \sin\,(36^\circ - A) + \sin\,(72^\circ + A) = \sin\,(36^\circ + A) + \sin\,(72^\circ - A)$,

which is *Euler's formula.*

Again, $\text{Sin}\,54^\circ = \dfrac{\sqrt{5}+1}{4}$; $\text{Sin}\,18^\circ = \dfrac{\sqrt{5}-1}{4}$; Art. 58. (3). (2).

$$\left.\begin{aligned} \text{and } 2\sin 54^\circ . \cos A &= \sin\,(54^\circ + A) + \sin\,(54^\circ - A) \\ 2\sin 18^\circ . \cos A &= \sin\,(18^\circ + A) + \sin\,(18^\circ - A) \end{aligned}\right\}$$

Art. 51. (1).

(4). $\therefore$ $\text{Cos}\,A$, or $\text{Sin}\,(90^\circ - A)$,

$$= \sin\,(54^\circ + A) + \sin\,(54^\circ - A) - \sin\,(18^\circ + A) - \sin\,(18^\circ - A),$$

which is *Legendre's formula.*

This formula might have been deduced from Euler's by writing $90^\circ - A$ for A.

Ex. To exemplify the use of these formulæ, we have from (4) by making $A = 13^\circ$,

$$\text{Cos}\,13^\circ = \sin 67^\circ + \sin 41^\circ - \sin 31^\circ - \sin 5^\circ.$$

Now the tables give for the quantities in the second member of the equation

$$9205{\cdot}049 + 6560{\cdot}590 - 5150{\cdot}381 - 871{\cdot}557,$$

$$\text{which} = 9743{\cdot}701,$$

the quantity given by the tables as the cosine of 13°.

Since, therefore, these quantities satisfy the relation which we know ought to exist between the sines of 67°, 41°, 31°, 5°, and the cosine of 13°, we conclude that the values of these natural goniometric functions are correctly given by the tables.

10. The values of the goniometric functions having been thus calculated, multiplied by 10000, verified, and registered in tables, are called "Tables of natural sines, cosines, tangents, cotangents, secants, and cosecants." To find the *real* goniometric functions from these tables, we have merely to divide the tabular numbers by 10000; that is, to remove the decimal point four places to the left.

11. We shall now pass to the logarithmic sines, cosines, tangents, &c. of angles, by which, rather than by the natural goniometric functions, mathematical calculations are most frequently made.

A page from Sherwin's Logarithmic Tables, corrected in some places, is here subjoined, the *natural* cosines, tangents, secants, and cosecants being omitted. The column following the *N*. sines, which is marked *N*. *D*. 1″, will be explained in the next Appendix.

17 Degrees.

M	N. Sine.	N. D. 1″.	L. Sine.	Diff. 1 sec.	Co-secants.	L. Tan.	Diff. 1 sec.	L. Sec.	Diff. 1 sec.	Co-sines.	M
0	2923·717		9·4659353	68·833	10·5340647	9·4853390	75·283	10·0194037	6·433	9·9805963	60
1	2926·499	46·36	9·4663483	68·766	10·5336517	9·4857907	75·200	10·0194423	6·450	9·9805577	59
2	2929·280		9·4667609	68·683	10·5332391	9·4862419	75·150	10·0194810	6·450	9·9805190	58
3	2932·061		9·4671730	68·633	10·5328270	9·4866928	75·083	10·0195197	6·466	9·9804803	57
4	2934·842		9·4675848	68·533	10·5324152	9·4871433	75·000	10·0195585	6·466	9·9804415	56
5	2937·623	46·33	9·4679960	68·483	10·5320040	9·4875933	74·950	10·0195973	6·466	9·9804027	55
6	2940·403		9·4684069	68·400	10·5315931	9·4880430	74·900	10·0196361	6·483	9·9803639	54
7	2943·183		9·4688173	68·333	10·5311827	9·4884924	74·816	10·0196750	6·500	9·9803250	53
8	2945·963		9·4692273	68·266	10·5307727	9·4889413	74·750	10·0197140	6·500	9·9802860	52
9	2948·743		9·4696369	68·200	10·5303631	9·4893898	74·700	10·0197529	6·500	9·9802471	51
10	2951·522	46·31	9·4700461	68·116	10·5299539	9·4898380	74·633	10·0197919	6·516	9·9802081	50
11	2954·302		9·4704548	68·050	10·5295452	9·4902858	74·566	10·0198310	6·516	9·9801690	49
12	2957·081		9·4708631	67·983	10·5291369	9·4907332	74·616	10·0198701	6·516	9·9801299	48
13	2959·859		9·4712710	67·916	10·5287290	9·4911802	74·450	10·0199092	6·533	9·9800908	47
14	2962·638		9·4716785	67·850	10·5283215	9·4916269	74·366	10·0199484	6·533	9·9800516	46
15	2965·416	46·29	9·4720856	67·766	10·5279144	9·4920731	74·316	10·0199876	6·533	9·9800124	45
16	2968·194		9·4724922	67·716	10·5275078	9·4925190	74·260	10·0200268	6·550	9·9799732	44
17	2970·971		9·4728985	67·633	10·5271015	9·4929646	74·183	10·0200661	6·550	9·9799339	43
18	2973·749		9·4733043	67·566	10·5266957	9·4934097	74·133	10·0201054	6·566	9·9798946	42
19	2976·526		9·4737097	67·483	10·5262903	9·4938545	74·050	10·0201448	6·566	9·9798552	41
20	2979·303	46·28	9·4741146	67·433	10·5258854	9·4942988	74·016	10·0201842	6·566	9·9798158	40
21	2982·079		9·4745192	67·366	10·5254808	9·4947429	74·060	10·0202236	6·583	9·9797764	39
22	2984·856		9·4749234	67·283	10·5250766	9·4951865	73·883	10·0202631	6·600	9·9797369	38
23	2987·632		9·4753271	67·216	10·5246729	9·4956298	73·816	10·0203027	6·600	9·9796973	37
24	2990·408		9·4757304	67·166	10·5242696	9·4960727	73·416	10·0203422	6·600	9·9796578	36
25	2993·184	46·25	9·4761334	67·083	10·5238666	9·4965152	73·700	10·0203818	6·616	9·9796182	35
26	2995·959		9·4765359	67·016	10·5234641	9·4969574	73·616	10·0204215	6·616	9·9795785	34
27	2998·734		9·4769380	66·933	10·5230620	9·4973991	73·583	10·0204612	6·616	9·9795388	33
28	3001·509		9·4773396	66·883	10·5226604	9·4978406	73·500	10·0205009	6·633	9·9794991	32
29	3004·284		9·4777409	66·816	10·5222591	9·4982816	73·450	10·0205407	6·633	9·9794593	31
30	3007·058	46·23	9·4781418	66·750	10·5218582	9·4987223	73·383	10·0205805	6·648	9·9794195	30
31	3009·832		9·4785423	66·666	10·5214577	9·4991626	73·333	10·0206204	6·633	9·9793796	29
32	3012·606		9·4789423	66·616	10·5210577	9·4996026	73·266	10·0206602	6·666	9·9793398	28
33	3015·380		9·4793420	66·533	10·5206580	9·5000422	73·200	10·0207002	6·650	9·9792998	27
34	3018·153		9·4797412	66·483	10·5202588	9·5004814	73·150	10·0207401	6·683	9·9792599	26
35	3020·926	46·22	9·4801401	66·400	10·5198599	9·5009203	73·083	10·0207802	6·666	9·9792198	25
36	3023·699		9·4805385	66·350	10·5194615	9·5013588	73·016	10·0208202	6·683	9·9791798	24
37	3026·471		9·4809366	66·266	10·5190634	9·5017969	72·966	10·0208603	6·683	9·9791397	23
38	3029·244		9·4813342	66·216	10·5186658	9·5022347	72·900	10·0209004	6·700	9·9790996	22
39	3032·016		9·4817315	66·133	10·5182685	9·5026721	72·850	10·0209406	6·700	9·9790594	21
40	3034·788	46·20	9·4821283	66·083	10·5178717	9·5031092	72·783	10·0209808	6·716	9·9790192	20
41	3037·559		9·4825248	66·000	10·5174752	9·5035459	72·716	10·0210211	6·716	9·9789789	19
42	3040·331		9·4829208	65·950	10·5170792	9·5039822	72·666	10·0210614	6·716	9·9789386	18
43	3043·102		9·4833165	65·866	10·5166835	9·5044182	72·600	10·0211017	6·733	9·9788983	17
44	3045·872		9·4837117	65·816	10·5162883	9·5048538	72·550	10·0211421	6·733	9·9788579	16
45	3048·643	46·16	9·4841066	65·733	10·5158934	9·5052891	72·483	10·0211825	6·750	9·9788175	15
46	3051·413		9·4845010	65·683	10·5154990	9·5057240	72·433	10·0212230	6·750	9·9787770	14
47	3054·183		9·4848951	65·616	10·5151049	9·5061586	72·366	10·0212635	6·750	9·9787365	13
48	3056·953		9·4852888	65·533	10·5147112	9·5065928	72·316	10·0213040	6·766	9·9786960	12
49	3059·723		9·4856820	65·483	10·5143180	9·5070267	72·250	10·0213446	6·766	9·9786554	11
50	3062·492	46·15	9·4860749	65·416	10·5139251	9·5074602	72·183	10·0213852	6·783	9·9786148	10
51	3065·261		9·4864674	65·350	10·5135326	9·5078933	72·133	10·0214259	6·783	9·9785741	9
52	3068·030		9·4868595	65·283	10·5131405	9·5083261	72·083	10·0214666	6·783	9·9785334	8
53	3070·798		9·4872512	65·233	10·5127488	9·5087586	72·016	10·0215073	6·800	9·9784927	7
54	3073·566		9·4876426	65·150	10·5123574	9·5091907	71·950	10·0215481	6·800	9·9784519	6
55	3076·334	46·11	9·4880335	65·083	10·5119665	9·5096224	71·916	10·0215889	6·816	9·9784111	5
56	3079·102		9·4884240	65·003	10·5115760	9·5100539	71·833	10·0216298	6·816	9·9783702	4
57	3081·869		9·4888142	64·966	10·5111858	9·5104849	71·783	10·0216707	6·833	9·9783293	3
58	3084·636		9·4892040	64·900	10·5107960	9·5109156	71·733	10·0217117	6·816	9·9782883	2
59	3087·403		9·4895934	64·833	10·5104066	9·5113460	71·666	10·0217526	6·850	9·9782474	1
60	3090·170		9·4899824		10·5100176	9·5117760		10·0217937		9·9782063	0
M	Co-sines.				L. Sec.	Co-tans.		Co-secants.		L. Sine.	M

72 Degrees.

APPENDIX III.

ON THE LOGARITHMIC TABLES OF GONIOMETRIC FUNCTIONS.

1. **When** the trigonometric functions of angles are determined, their logarithms may be found from the tables of the logarithms of numbers. There are however methods by which the logarithms of the goniometric functions can be found independently.

2. *To find* $\mathrm{l}_{10}\sin\theta$, $\sin\theta$ *not being given.*

$$\mathrm{Sin}\,\theta=\theta.\left(1-\frac{\theta^2}{\pi^2}\right).\left(1-\frac{\theta^2}{2^2.\pi^2}\right).\left(1-\frac{\theta^2}{3^2.\pi^2}\right)\ldots.\text{Art. 141. p. 139.}$$

Making $\theta=\frac{m}{n}.\frac{\pi}{2}$, and taking the logarithms of both sides of this equation, we have

$$\therefore\ \mathrm{l}_{10}\sin\left(\frac{m}{n}.\frac{\pi}{2}\right)=\mathrm{l}_{10}\left(\frac{m}{n}.\frac{\pi}{2}\right)+\mathrm{l}_{10}\left(1-\frac{m^2}{2^2.n^2}\right)+\mathrm{l}_{10}\left(1-\frac{m^2}{4^2.n^2}\right)$$

$$+\,\mathrm{l}_{10}\left(1-\frac{m^2}{6^2.n^2}\right)+\ldots$$

$$=\mathrm{l}_{10}\left(\frac{m}{n}.\frac{\pi}{2}\right)+\mathrm{l}_{10}\left(\frac{2^2.n^2-m^2}{2^2.n^2}\right)$$

$$-\frac{1}{\mathrm{l}_e 10}.\left(\frac{m^2}{4^2.n^2}+\tfrac{1}{2}.\frac{m^4}{4^4.n^4}+\tfrac{1}{3}.\frac{m^6}{4^6.n^6}+\ldots\right)$$

$$-\frac{1}{\mathrm{l}_e 10}.\left(\frac{m^2}{6^2.n^2}+\tfrac{1}{2}.\frac{m^4}{6^4.n^4}+\tfrac{1}{3}.\frac{m^6}{6^6.n^6}+\ldots\right)$$

$$-\frac{1}{\mathrm{l}_e 10}.\left(\frac{m^2}{8^2.n^2}+\tfrac{1}{2}.\frac{m^4}{8^4.n^4}+\tfrac{1}{3}.\frac{m^6}{8^6.n^6}+\ldots\right)$$

$$-\ \&\text{c.}$$

($\mathrm{l}_{10}\left(1-\frac{m^2}{4^2.n^2}\right)$, $\mathrm{l}_{10}\left(1-\frac{m^2}{6^2.n^2}\right)$, &c. being expanded by (ii.), p. 151.)

Now $l_{10}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right) = l_{10}m + l_{10}\pi - l_{10}n - l_{10}2,$

and $$l_{10}\frac{2^2.n^2 - m^2}{2^2.n^2} = l_{10}\{(2n+m).(2n-m)\} - l_{10}2^2n^2$$
$$= l_{10}(2n+m) + l_{10}(2n-m) - 2.\{l_{10}2 + l_{10}n\};$$

$$\therefore l_{10}\sin\left(\frac{m}{n}\cdot\frac{\pi}{2}\right) = l_{10}m + l_{10}(2n+m) + l_{10}(2n-m) + l_{10}\pi - 3.\{l_{10}n + l_{10}2\}$$
$$-\frac{1}{l_{,}10}\times\left\{\begin{array}{l} \left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots\right).\frac{m^2}{n^2} \\ +\frac{1}{2}.\left(\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\ldots\right).\frac{m^4}{n^4} \\ +\frac{1}{3}.\left(\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\ldots\right).\frac{m^6}{n^6} \\ +\&c. \end{array}\right\}.$$

By giving m and n different values, the logarithmic sines of all angles may be found by this formula.

3. In like manner from the series

$$\text{Cos}\,\theta = \left(1-\frac{2^2.\theta^2}{\pi^2}\right).\left(1-\frac{2^2.\theta^2}{3^2.\pi^2}\right).\left(1-\frac{2^2.\theta^2}{5^2.\pi^2}\right)\ldots$$ Art. 141. p. 139.

we get the formula

$$l_{10}\cos\left(\frac{m}{n}\cdot\frac{\pi}{2}\right) = l_{10}(n+m) + l_{10}(n-m) - 2l_{10}n$$
$$-\frac{1}{l_{,}10}\times\left\{\begin{array}{l} \left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right).\frac{m^2}{n^2} \\ +\frac{1}{2}.\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ldots\right).\frac{m^4}{n^4} \\ +\frac{1}{3}.\left(\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ldots\right).\frac{m^6}{n^6} \\ +\&c. \end{array}\right\}.$$

4. The logarithms of the sines and cosines having been thus determined, those of

$$\operatorname{Tan}\theta = \frac{\sin\theta}{\cos\theta},\quad \operatorname{Sec}\theta = \frac{1}{\cos\theta},\quad \operatorname{Cot}\theta = \frac{\cos\theta}{\sin\theta},\quad \operatorname{Cosec}\theta = \frac{1}{\sin\theta},$$

may be severally found.

5. Since all sines and cosines are, generally, less than 1, their logarithms to base 10 are negative. In order to avoid the inconvenience of printing negative characteristics, the logarithms to base of 10 of *all* goniometrical quantities have been increased by 10, and the resulting numbers being registered are *called* "Tables of logarithmic sines, cosines, &c."

Hence if any one of these *tabular* logarithmic quantities be given, by subtracting 10 from it we get the *real* logarithm of the goniometric quantity.

These tabular logarithmic quantities we shall indicate by the letter L; thus the tabular logarithmic sine of A, or $10 + l_{10} \sin A$, will be written $L \sin A$.

6. The common Logarithmic Tables of goniometric functions are calculated for angles which differ from one another by one minute. If, besides degrees and minutes, the angle contain some seconds, its tabular logarithmic function may, with certain exceptions, be found on the principle proved in the next Article.

7. *The increments of tabular logarithmic functions of angles vary, except in certain cases, as the increment of the angle.*

Let the angle A receive the increments $\delta A''$, and $60''$, successively.

Then $$\operatorname{Sin}(A + \delta A'') = \sin A \,.\left\{1 + \frac{\sin(A + \delta A'') - \sin A}{\sin A}\right\}$$

$$= \sin A \,.\left\{1 + \frac{\cos A \,.\, \sin \delta A''}{\sin A}\right\},$$

unless $A = 90^0$ nearly; Art. 60. Cor.

$$\therefore\ \mathrm{l}_{10}\sin(A+\delta A'') = \mathrm{l}_{10}\sin A + \mathrm{l}_{10}(1+\cot A\,.\,\sin\delta A'');$$

$$\therefore\ \{10+\mathrm{l}_{10}\sin(A+\delta A'')\} - \{10+\mathrm{l}_{10}\sin A\} = \mathrm{l}_{10}(1+\cot A\,.\,\sin\delta A'');$$

$$\therefore\ L\sin(A+\delta A'') - L\sin A$$

$$= \frac{1}{\mathrm{l}_{\epsilon}10}\,.\,\{\cot A\,.\,\sin\delta A'' - \tfrac{1}{2}(\cot A)^2\,.\,(\sin\delta A'')^2 + \ldots\}\ \text{App. I. (ii.)}$$

$$= \frac{1}{\mathrm{l}_{\epsilon}10}\,.\,\cot A\,.\,\sin\delta A'',$$

neglecting the higher powers of $\cot A\,.\,\sin\delta A''$, which may be done unless $A = 2n\,.\,90^0$ nearly.

And writing $60''$ for $\delta A''$ in this equation, it becomes

$$L\sin(A+60'') - L\sin A = \frac{1}{\mathrm{l}_{\epsilon}10}\,.\,\cot A\,.\,\sin 60'';$$

$$\therefore\ \frac{L\sin(A+\delta A'') - L\sin A}{L\sin(A+60'') - L\sin A} = \frac{\sin\delta A''}{\sin 60''} = \frac{\delta A}{60}.\quad \text{Art. 116. Cor.}$$

And in a manner exactly similar it may be shewn that for the Cosine, Tangent, &c., of an angle, the increment of the tabular logarithm varies as the increment of the angle, except in those cases mentioned in the Corollaries to Arts. 60, 61, 62, 63.

8. *To explain the meaning and use of the columns of differences for one second* (Diff. 1″), *which are placed after the columns of logarithmic functions.*

If $\delta A''$ become $1''$, the equation of Art. 7. becomes,

$$L\sin(A+1'') - L\sin A = \{L(\sin A + 60'') - L\sin A\}\,.\,\frac{1}{60},$$

which is the difference of $L\sin A$ corresponding to one second.

Now, if this quantity be computed and registered, when δA is given, $L(\sin A + \delta A'') - L\sin A$ may be determined by merely multiplying this registered difference by δA,—and when $L \sin (A + \delta A'')$ is given, δA may be found by dividing $L \sin (A + \delta A'') - L \sin A$ by this difference. For

$$L \sin (A + \delta A'') - L \sin A = \{L \sin (A + 60'') - L \sin A\} . \frac{\delta A}{60}$$

$$= \frac{L \sin (A + 60'') - L \sin A}{60} . \delta A,$$

$$\text{and } \delta A = \frac{L \sin (A + \delta A'') - L \sin A}{\frac{1}{60} . \{L \sin (A + 60'') - L \sin A\}} .$$

$$\text{Thus, } \quad L \sin 17^0, 1' = 9{\cdot}4663483$$

$$L \sin 17^0 \quad = 9{\cdot}4659353$$

$$\therefore \ L \sin 17^0\, 1' - L \sin 17^0 = \ {\cdot}0004130$$

Now $\frac{4130}{60} = 68{\cdot}833$, which is the quantity put down in the Tables as the difference for one second to the sines of angles between 17° and 17°, 1′.

The significant part of the difference is considered as a whole number, or the *real* difference is multiplied by 10^7, to avoid the necessity of printing the three or four cyphers which nearly in every case precede the significant part of the difference[*].

9. Ex. 1. *To find* L sin 17°, 14′, 12″.

$$L \sin 17^0, 14', \quad = 9{\cdot}4716785$$

$$\text{Now Diff. for } 1'' = 67{\cdot}850$$

$$\therefore \text{ Diff. for } 12'' = 814{\cdot}200 = \quad 814{\cdot}2$$

$$\therefore \ L \sin 17^0, 14', 12'' = 9{\cdot}4717599.$$

---

[*] The column of differences for the natural sines &c. of angles are computed after a manner similar to this, and the differences themselves are all multiplied by 1000, for the sake of leaving out the cyphers immediately following the decimal point.

Ex. 2. *If* $\text{L}\sin\text{A} = 9{\cdot}4685537$, *required* A.

$$\begin{array}{rl} L\sin A = & 9{\cdot}4685537 \\ L\sin 17^{0}, 6' = & \underline{9{\cdot}4684069} \\ \text{Diff.} = & \phantom{9{\cdot}468}1468 \end{array}$$

Now *Diff. for* 1″ is in this case 68·400, and $\dfrac{1468}{68{\cdot}400} = 21{\cdot}46$;

$$\therefore\ A = 17^{0}, 6', 21''{\cdot}46.$$

Ex. 3. *If* $\text{L}\cos\text{A} = 9{\cdot}9784328$, *required* A.

In this case, because the increase of the angle is attended by the decrease of the *L* cosine, Art. 61, Cor. 2, we subtract the given *L* cosine from that in the tables which is next greater than it.

$$\begin{array}{rl} \text{Now }L\cos 17^{0}, 54' = & 9{\cdot}9784519 \\ L\cos A \qquad = & \underline{9{\cdot}9784328} \\ \text{Diff.} = & \phantom{9{\cdot}9784}191 \end{array}$$

Now *Diff. for* 1″ in this case is 6·800,

$$\text{and } \frac{191}{6{\cdot}800} = 28{\cdot}088 = 28{\cdot}09 \text{ nearly};$$

$$\therefore\ A = 17^{0}, 54', 28''{\cdot}09.$$

Ex. 4. *Required the* L *cosine of* 72°, 5′, 8″.

By the Tables, p. 173.

$$\begin{array}{lrl} L\cos 72^{0}, 5', & = 9{\cdot}4880335 & \text{, and } \textit{Diff. for } 1'' = 65{\cdot}15; \\ \therefore \text{Diff. for} \quad 8'' = & 521{\cdot}2 & \\ \hline \therefore\ L\cos 72^{0}, 5', 8'' = & 9{\cdot}4879814. & \end{array}$$

The difference for the seconds being in this case *subtracted* from $L\cos 72^{0}, 5'$. Art. 60.

Note. It may here be observed, that the difference for additional seconds must be *added* for *L* sines, *L* tangents, and *L* secants, Arts. 60, 62, 63; and *subtracted* for *L* cosines, Art. 60, *L* cotangents and *L* cosecants.

10. *To shew that the same columns of "Differences for* $1''$" *serve for* L sin A *and* L cosec A, *for* L cos A *and* L sec A, *and for* L tan A *and* L cot A.

$$\text{For } \sin A = \frac{1}{\operatorname{cosec} A};$$

$$\therefore\ \mathrm{l}_{10} \sin A = -\,\mathrm{l}_{10} \operatorname{cosec} A,$$

$$\therefore\ L \sin A = 10 + \mathrm{l}_{10} \sin A = 20 - (10 + \mathrm{l}_{10} \operatorname{cosec} A) = 20 - L \operatorname{cosec} A.$$

$$\text{Similarly, } L \sin (A + 1'') = 20 - L \operatorname{cosec} (A + 1'')$$

$$\therefore\ L \sin (A + 1'') - L \sin A = -\left\{ L \operatorname{cosec} (A + 1'') - L \operatorname{cosec} A \right\}.$$

Hence a column of "differences for $1''$" is printed *between* the columns of logarithmic sines and cosecants; serving to the former as a column of *increments* for $1''$, and to the latter as a column of *decrements* for $1''$.

In like manner it may be shewn that

$$L \cos (A + 1'') - L \cos A = -\left\{ L \sec (A + 1'') - L \sec A \right\}$$

$$L \tan (A + 1'') - L \tan A = -\left\{ L \cot (A + 1'') - L \cot A \right\}.$$

Wherefore the columns of cosines and secants have the same differences for $1''$, as also have the tangents and cotangents: and it is to be observed that these differences serve respectively as increments to the secants, (Art. 62.), and to the tangents (Art, 63.), and as decrements to the cosines (Art. 61.), and to the cotangents.

11. Before the increment of a tabular logarithmic function of an angle can be determined from the *small* given increment of the angle, or conversely, the two following conditions must be fulfilled.

I. The logarithmic increment of the function must, in that particular case, vary as the increment of the angle.

II. The increment of the logarithmic function must not be an exceedingly small quantity.

Thus, if it were required to determine $L \sin 89^\circ, 40', 3''$, from Tables in which L sines were registered for all angles from 0° to 90° which differed from one another by $1'$, we should find

$$L \sin 89^\circ, 41' = 9{\cdot}9999934$$

$$L \sin 89^\circ, 40' = 9{\cdot}9999927$$

$$\therefore \text{ Difference for } 60'' = \qquad 7$$

Wherefore, if even the second of these conditions held for the L sines of angles about $89^\circ, 40'$ in magnitude, (which it does not, Art. 60. Note), yet a difference of 1 in the L sine would produce a difference of only $\frac{60''}{7}$, or $9''$ nearly, in the angle;—and therefore any increment less than $8''$ of $89^\circ, 40'$, would produce no change in the first seven figures following the decimal point of $L \sin 89^\circ, 40'$.

Note. What has here been said with respect to the logarithmic, is equally true of the natural goniometric functions.

12. It has been observed, App. III. 5, that the *real* logarithm of a goniometrical quantity is obtained by subtracting 10 from the tabular logarithm; we shall now proceed

To establish a general rule for supplying the TENS *when the tabular logarithms of goniometrical functions are used.*

Let $(\text{Cos } A)^n = a \,.\, (\sin B)^m \,.\, \tan C)^p$ be any trigonometrical formula adapted to logarithmic computation, App. I. 25; B and C being given angles, and A being required.

$$\text{Then } n \,.\, l_{10} \cos A = l_{10}\, a + m \,.\, l_{10} \sin B + p \,.\, l_{10} \tan C;$$

$$\therefore\ n \,.\, (10 + l_{10} \cos A) - n \,.\, 10 = l_{10}\, a + m \,.\, (10 + l_{10} \sin B) - m \,.\, 10$$
$$+ p \,.\, (10 + l_{10} \tan C) - p \,.\, 10;$$

$$\therefore\ n \,.\, L \cos A = \mathrm{l}_{10} a + m \,.\, L \sin B + p \,.\, L \tan C + \{\, n - (m+p) \,\} \,.\, 10.$$

And here *n*, *m*, *p* may be whole or fractional.

Whence the rule, *Add to the second member of the equation as many tens as the number of times the tabular logarithms of the goniometrical functions have been taken in the first member exceeds the number of times they have been taken in the second member of the equation.*

Ex. 1. $(\mathrm{Tan}\, A)^5 = \cos B \,.\, (\cos C)^2;$

$$\therefore\ 5 \,.\, L \tan A = L \cos B + 2 \,.\, L \cos C + \{\, 5 - (1 + 2) \,\} \,.\, 10$$

$$= L \cos B + 2 \,.\, L \cos C + 20.$$

Ex. 2. $(\mathrm{Tan}\, A)^2 . (\sin B)^6 = \frac{a}{b} \,.\, (\sec C)^4,$

$$2 \,.\, L \tan A + 6 \,.\, L \sin B = \mathrm{l}_{10} a - \mathrm{l}_{10} b + 4 \,.\, L \sec C + \{\, (2 + 6) - 4 \,\} \,.\, 10$$

$$= \mathrm{l}_{10} a - \mathrm{l}_{10} b + 4 \,.\, L \sec C + 40.$$

Ex. 3. $(\mathrm{Tan}\, A)^{\frac{3}{2}} = \dfrac{2 \,.\, (\sin B)^{\frac{5}{2}}}{(\cos C)^{\frac{2}{3}}};$

$$\therefore\ \frac{3}{2} \,.\, L \tan A = \mathrm{l}_{10} 2 + \frac{5}{2} \,.\, L \sin B - \frac{2}{3} \,.\, L \cos C + \left\{ \frac{3}{2} - \left(\frac{5}{2} - \frac{2}{3} \right) \right\} 10$$

$$= \mathrm{l}_{10} 2 + \frac{5}{2} \,.\, L \sin B - \frac{2}{3} \,.\, L \cos C - \tfrac{1}{3} \,.\, 10.$$

Had both sides of this equation been raised to the sixth power, the fractional indices would have disappeared, and the value of tan *A* would practically have been determined much more easily.

13. Ex. ***Required the value of* A *from the equation***

$$(\text{Cos}\,A)^2 = \frac{36{\cdot}61468}{11{\cdot}9}\,.\,\sin 17^\circ, 9', 4''.$$

$$2\,.\,L\cos A = \text{l}_{10}36{\cdot}61468 - \text{l}_{10}11{\cdot}9 + L\sin 17^\circ, 9', 4'' + 10\,;$$

$$\therefore\;L\cos A + \tfrac{1}{2}\,\text{l}_{10}36{\cdot}61468 - \text{l}_{10}11{\cdot}9 + L\sin 17^\circ, 9', 4'' + 10)$$

$$\begin{array}{lrr}
 & \text{l}_{10}36{\cdot}614 & = 1{\cdot}5636472 \\
\text{Part for} & 6 = & 71 \\
\text{Part for} & 8 = & 10 \\
\hline
\therefore & \text{l}_{10}36{\cdot}61468 & = 1{\cdot}5636553 \\
 & \text{l}_{10}11{\cdot}9 & = 1{\cdot}0755470 \\
\hline
 & & {\cdot}4881083 \\
 & L\sin 17^\circ, 9' & = 9{\cdot}4696369 \\
\text{Part } (4 \times 68{\cdot}200) \text{ for} & 4'' = & 273 \\
 & \text{Add} & 10{\cdot} \\
\hline
 & 2) & 19{\cdot}9577725 \\
\hline
 & & 9.9788862
\end{array}$$

The next higher L cosine in the Tables is that of 17°, 43′, which is

$$\begin{array}{rr}
 & 9{\cdot}9788983 \\
\text{Subtract} & 9.9788862 \\
\hline
\text{Diff.} = & 121
\end{array}$$

Now ***Diff. for*** 1″ in this case is 6·733, and $\dfrac{121}{6{\cdot}733} = 17{\cdot}97\,;$

$$\therefore\;A = 17^\circ, 43', 17''{\cdot}97.$$

14. We shall, lastly, explain the methods used to determine small angles from their logarithmic functions, and conversely.

When an angle receives a small increment, the Differential Calculus enables us to determine without difficulty the consequent increase of the goniometric function.

Required the increase of L $\sin\theta$ *arising from* θ *receiving a small increment* $\delta\theta$.

By TAYLOR'S THEOREM,

$$L\sin(\theta+\delta\theta) = L\sin\theta + d_\theta L\sin\theta \,.\, \delta\theta + d_\theta^2 L\sin\theta \,.\, \frac{(\delta\theta)^2}{1\,.\,2} + \ldots$$

Now $L\sin\theta = 10 + \mathrm{l}_{10}\sin\theta$;

$$\therefore\ d_\theta L\sin\theta = \frac{1}{\mathrm{l}_\epsilon 10}\cdot\frac{\cos\theta}{\sin\theta} = \frac{1}{\mathrm{l}_{10}10}\,.\,\cot\theta,$$

$$d_\theta^2 L\sin\theta = -\frac{1}{\mathrm{l}_\epsilon 10}\,.\,(\mathrm{cosec}\,\theta)^2;$$

$$\therefore\ L\sin(\theta+\delta\theta) - L\sin\theta$$

$$= \frac{1}{\mathrm{l}_\epsilon 10}\,.\,\cot\theta\,.\,\delta\theta - \frac{1}{\mathrm{l}_\epsilon 10}\,.\,(\mathrm{cosec}\,\theta)^2\,.\,\frac{(\delta\theta)^2}{1\,.\,2} + \ldots\ldots$$

15. Now we know that if θ be very small, $\mathrm{cosec}\,\theta$ is very large, and therefore unless $\delta\theta$ be exceedingly small also, the second difference, $-\frac{1}{\mathrm{l}_\epsilon 10}\,.\,(\mathrm{cosec}\,\theta)^2\,.\,\frac{(\delta\theta)^2}{1\,.\,2}$, is of such magnitude that it cannot be neglected in comparison with the first difference, $\frac{1}{\mathrm{l}_\epsilon 10}\,.\,\cot\theta\,.\,\delta\theta$. In this case therefore, since the increment of $L\sin\theta$ does not vary as $\delta\theta$ the simple power of the increment of θ, the quantity to be added to $L\sin\theta$ for a small increment of θ cannot be obtained by a simple proportion,

but will have to be determined approximately by the tedious process of computing the first two terms of the series which gives the value of the increment of $L\sin\theta$.

There are three methods of escaping this inconvenience.

16. The first is, to construct tables for the first few degrees to intervals of a second, instead of to intervals of a minute, as the tables in ordinary use are constructed. By this means $\delta\theta$ must be less than one second, and $L\sin(\theta+\delta\theta)$ may be roughly computed to decimal parts of a second by neglecting the second term of series of Art. 14,—in which case,

$$L\ \sin(\theta+\delta\theta) - L\ \sin\theta = \frac{1}{\mathrm{l}_{\epsilon}10}\cdot\cot\theta\,.\,\delta\theta\,;$$

Therefore, for any particular value of θ,

$$\Delta L\ \sin\theta \propto \delta\theta.$$

17. Second method. By the following formulæ which are given by Maskelyne in his Introduction to Taylor's Logarithms, we can, with the aid of tables of L sines and L cosines calculated to every second for a few degrees, determine a small angle very accurately to decimal parts of a second from its L sine, and conversely.

$$\operatorname{Sin}\theta = \theta - \frac{\theta^3}{2\,.\,3} + \ldots \qquad = \theta - \frac{\theta^3}{2\,.\,3} \quad \text{nearly.}$$

$$\operatorname{Cos}\theta = 1 - \frac{\theta^2}{1\,.\,2} + \ldots \qquad = 1 - \frac{\theta^2}{2} \quad \text{nearly.}$$

$$\therefore\ \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{2\,.\,3} = \left(1 - \frac{\theta^2}{2}\right)^{\frac{1}{3}} \text{ nearly, } = (\cos\theta)^{\frac{1}{3}}.$$

Let θ be an angle containing n'',

$$\text{then } n = \frac{\theta}{\sin 1''}, \text{ or } \theta = n\,.\sin 1'';$$

$$\therefore\ \operatorname{Sin}\theta = n\sin 1''.\,(\cos\theta)^{\frac{1}{3}},$$

Writing n'' for θ, and taking the logarithms,

$$L \sin n'' = \mathrm{l}_{10} n + L \sin 1'' + \tfrac{1}{3} L \cos n'' - \tfrac{1}{3} . 10\,;$$

$$\therefore\ L \sin n'' = \mathrm{l}_{10} n + L \sin 1'' - \tfrac{1}{3} . (10 - L \cos n'') \ldots\ldots\ldots \text{(i.)},$$

$$\mathrm{l}_{10} n = L \sin n'' + \tfrac{1}{3} (10 - L \cos n'') - L \sin 1'' \ldots\ldots \text{(ii.)}$$

Def. The quantity $10 - L \cos n''$ is called the arithmetic complement of $L \cos n''$.

18. It is to be remarked, that in using these formulæ to determine $L \sin n''$ when n is given, or conversely, we may on the second member of the equations take from the tables an *approximate* value of $L \cos n''$ without affecting the result,—because the variation of $L \cos n''$ is exceedingly small when n is small, as may easily be shewn[*]. The whole matter will be rendered more clear by an example.

Ex. *If* L sin n″ = 7·3217783, *required* n.

Now Taylor's tables give,

$L \sin 7', 12''$	7·3210583	$L \cos 7', 12''$	9·9999990
$L \sin 7', 13''$	7·3220624	$L \cos 7', 13''$	9·9999990

Therefore the angle is 7′, 12″ nearly; and 7′, 12″ is the approximate value of the angle we must take in the second member of (ii) in Art. 17.

[*] $$\operatorname{Cos} \theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} \ldots\ldots \quad = 1 - \frac{\theta^2}{2} \text{ nearly};$$

$$\therefore\ \mathrm{l}_{10} \cos \theta, \quad \text{or } L \cos \theta - 10, \quad = - \frac{1}{\mathrm{l}_\epsilon 10} \cdot \left\{ \frac{\theta^2}{2} + \tfrac{1}{2} \cdot \left(\frac{\theta^2}{2} \right)^2 + \ldots \right\};$$

$$\therefore\ d_\theta (L \cos \theta - 10) = - \frac{1}{\mathrm{l}_\epsilon 10} \cdot (\theta + \frac{\theta^3}{2} + \ldots),$$

a very small quantity, if θ be an angle of a few seconds only.

Now by (ii.),

$$L \sin n'' = 7{\cdot}3217783,$$

$$\left.\begin{array}{l}\tfrac{1}{3}\,(10 - L\cos 7',\, 12'')\\ \text{or } \tfrac{1}{3} \times {\cdot}0000010,\end{array}\right\} = \begin{array}{r}{\cdot}0000003\\ \hline 7{\cdot}3217786\end{array}$$

$$L \sin 1'' = 4{\cdot}6855749$$

$$\therefore\ \mathrm{l}_{10} n = 2{\cdot}6362037 = \mathrm{l}_{10} 432{\cdot}723\,;$$

$\therefore\ n = 432{\cdot}723$, or the angle required is $7',\, 12''{\cdot}723$.

19. In like manner a formula may be established for finding $L \tan n''$ from n, and conversely.

$$\operatorname{Tan} \theta = \frac{\sin \theta}{\cos \theta} = \frac{\theta - \dfrac{\theta^3}{2\,.\,3}}{1 - \dfrac{\theta^2}{2}} \text{ nearly}\,;$$

$$\therefore\ \frac{\tan \theta}{\theta} = \frac{1 - \dfrac{\theta^2}{2\,.\,3}}{1 - \dfrac{\theta^2}{2}} = \frac{\left(1 - \dfrac{\theta^2}{2}\right)^{\frac{1}{3}}}{1 - \dfrac{\theta^2}{2}} = \left(1 - \frac{\theta^2}{2}\right)^{-\frac{2}{3}} = (\cos \theta)^{-\frac{2}{3}}\,;$$

$$\frac{\tan n''}{n\,.\,\sin 1''} = (\cos n'')^{-\frac{2}{3}}\,;$$

$$\therefore\ L \tan n'' = \mathrm{l}_{10} n + L \sin 1'' - \frac{2}{3}\,.\,L \cos n'' + \frac{2}{3}\,.\,10\,;$$

$$\therefore\ L \tan n'' = \mathrm{l}_{10} n + L \sin 1'' + \frac{2}{3}\,.\,(10 - L \cos n'') \ldots\ldots\ldots\ldots \text{(iii.)}$$

$$\text{And } \mathrm{l}_{10} n = L \tan n'' - L \sin 1'' - \frac{2}{3}\,.\,(10 - L \cos n'') \ldots\ldots\ldots \text{(iv.)}$$

20. ***Delambre's Tables.*** The third method is to construct tables up to a degree, which give $\mathrm{l}_{10}\dfrac{\sin\theta}{\theta}$, (or tables which give $\mathrm{l}_{10}\dfrac{\sin\theta}{\theta}+L\sin 1''$), for angles differing by one second. Such tables are, we believe, printed in no collection except those of Callet; they may, however, be easily constructed for use in the following manner:

Let θ be an angle of n seconds; $\therefore\ \theta = n\,.\sin 1''$.

$$\text{Then } \mathrm{l}_{10}\frac{\sin\theta}{\theta} = \mathrm{l}_{10}\frac{\sin n''}{n\,.\sin 1''} = \mathrm{l}_{10}\sin n'' - \mathrm{l}_{10}n - \mathrm{l}_{10}\sin 1''$$

$$= L\sin n'' - \mathrm{l}_{10}n - L\sin 1'';$$

$$\therefore\ \mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1'' = L\sin n'' - \mathrm{l}_{10}n.$$

Similarly, if θ be an angle of n minutes, we have

$$\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1' = L\sin n' - \mathrm{l}_{10}n.$$

21. *To determine the sine of a given small angle, or conversely, from Delambre's Tables.*

$$\text{Since}\quad \text{Sin}\,\theta = {}^{\sin\theta}\,.\,\theta = \frac{\sin\theta}{\theta}\,.\,n\,.\sin 1'';$$

$$\therefore\ L\sin\theta = \left(\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1''\right) + \mathrm{l}_{10}n;$$

$$\therefore\ L\sin n'' = \left(\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1''\right) + \mathrm{l}_{10}n\ldots\ldots\ldots\ldots\text{(i.)}$$

$$\text{And}\quad \mathrm{l}_{10}n = L\sin n'' - \left(\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1''\right)\ldots\ldots\ldots\text{(ii.)}$$

The most convenient tables are evidently those which give $\left(\mathrm{l}_{10}\dfrac{\sin\theta}{\theta} + L\sin 1''\right)$ for every second.

22. Ex. 1. *To determine* L sin n″ *by Delambre's Tables.*

Since, as shewn below*, $\mathrm{l}_{10}\dfrac{\sin\theta}{\theta} + L\sin 1''$ increases very slowly, we may take for $\mathrm{l}_{10}\dfrac{\sin\theta}{\theta} + L\sin 1''$, the quantity given in the tables for the angle containing that number of seconds which is the nearest integer to the given number (n), and by adding $\mathrm{l}_{10}n$ to this quantity we obtain $L\sin n''$.

Thus; *If* n = 546·25, *required* L sin 546″·25.

By Taylor's Tables. $L\sin 546''$, or $L\sin 9', 6''$, is 7·4227670; and $\mathrm{l}_{10}546 = 2{\cdot}7371926$.

Therefore, when θ is an angle of 546″,

$$\mathrm{l}_{10}\frac{\sin\theta}{\theta} + L\sin 1'' = L\sin n'' - \mathrm{l}_{10}n \quad \text{Art. 20.}$$

$$= 7{\cdot}4227670 - 2{\cdot}7371926$$

$$= 4{\cdot}6855744,$$

the quantity corresponding to the angle 546″ which would be given in the tables of $\mathrm{l}_{10}\dfrac{\sin\theta}{\theta} + L\sin 1''$.

∴ By (i.) Art. 21,

$$L\sin 546''{\cdot}25 = 4{\cdot}6855744 + \mathrm{l}_{10}546{\cdot}25$$

$$= 4{\cdot}6855744 + 2{\cdot}7373914$$

$$= 7{\cdot}4229658.$$

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* For $\dfrac{\sin\theta}{\theta} = 1 - \dfrac{\theta^2}{1.2.3} + \dfrac{\theta^4}{1.2.3.4.5} - \ldots = 1 - \dfrac{\theta^2}{6}$ nearly;

$\therefore\ \mathrm{l}_{10}\dfrac{\sin\theta}{\theta} = -\dfrac{1}{\mathrm{l}_{\varepsilon}10}\cdot\left\{\dfrac{\theta^2}{6} + \tfrac{1}{2}\left(\dfrac{\theta^2}{6}\right)^2 + \ldots\right\}$; $\therefore\ d_\theta \mathrm{l}_{10}\dfrac{\sin\theta}{\theta} = -\dfrac{1}{\mathrm{l}_{\varepsilon}10}\cdot\left(\dfrac{\theta}{3} + \dfrac{2\theta^3}{6^2} + \ldots\right)$, a very small quantity, if θ be an angle of a few minutes only.

Ex. 2. *To determine* n *where* L sin n″ *is given.*

By referring to the tables of *L* sines in common use we see what integral number of seconds is contained in the angle whose *L* sine is next below the proposed quantity: let this number be *m*. Then substituting in (ii.) Art. 21, the quantity given for $l_{10}\frac{\sin\theta}{\theta} + L\sin 1''$ corresponding to the angle m'', we obtain a near approximation to the value of *n*.

Thus, *If* L sin n″ = 7·4230612, *required* n.

Referring to Taylor's Tables, we find

$$L\sin 546'' = 7{\cdot}4227670,$$

$$L\sin 547'' = 7{\cdot}4235617\,;$$

∴ the value of *m* in this case is 546.

Now, as we have seen in the last example, the value of $l_{10}\frac{\sin\theta}{\theta} + L\sin 1''$ for an angle 546″ is 4·6855744.

Therefore by (ii.), Art. 19,

$$l_{10}n = 7{\cdot}4230612 - 4{\cdot}6855744$$

$$= 2{\cdot}7374868 = l_{10}546{\cdot}37\,;$$

∴ 546″·37 is the angle sought.

EXAMPLES.

1. Prove that $45^0, 15', 20'' = 50^g, 28^{\backprime}, 39^{\backprime\backprime}, 50^{\backprime\backprime\backprime}$; $11^0, 15', 37'' = 11^g, 40^{\backprime}, 3^{\backprime\backprime}, 9^{\backprime\backprime\backprime}$; $18^0, 12' = 20^g, 20^{\backprime}$; $\dfrac{180^0}{\sqrt{3}} = 115^g, 47^{\backprime}$; $1^{\backprime\backprime}, 2^{\backprime\backprime\backprime}, 3^{\backprime\backprime\backprime\backprime} = 19''', 50''''\cdot 0779$.

2. The Complements of $17^0, 36', 43''$; $29^0, 27', 6''\cdot 32$; and $216^0, 45'$; are $72^0, 23', 17''$; $60^0, 32', 53''\cdot 68$; and $-126^0, 45'$.

3. The Supplements of $37^0, 4', 3''$; $115^0, 13', 24''\cdot 66$; and $226^0, 14', 17''$; are $142^0, 55', 57''$; $64^0, 46', 35''\cdot 34$; and $-(46^0, 14', 17'')$.

4. If $\text{Cot}\, A = \dfrac{3}{2}$, find the values of $\sin A$, $\cos A$, $\text{cosec}\, A$, $\text{versin}\, A$, and $\sec A$.

5. Prove the formulæ,

(1). $(\text{Sec}\, A \,.\, \text{cosec}\, A)^2 = (\sec A)^2 + (\text{cosec}\, A)^2$.

(2). $(\cot A \,.\, \cos A)^2 = (\cot A)^2 - (\cos A)^2$.

(3). $\text{Cos}\, A = \dfrac{\cot A}{\sqrt{\{1 + (\cot A)^2\}}}$. (4). $\text{Versin}\, A = \dfrac{\sec A - 1}{\sec A}$.

(5). $\text{Sin}\, A \,.\, \cos A = \dfrac{1}{\tan A + \cot A}$.

6. (1). If $(\tan A)^2 + 4(\sin A)^2 = 6$; $A = 60^0$.

(2). If $m = \tan A + \sin A$, and $n = \tan A - \sin A$; $\text{Cos}\, A = \dfrac{m-n}{m+n}$.

(3). If $m \,.\, \sin A = n \,.\, \cos A$; $\text{Sin}\, A = \pm \dfrac{n}{\sqrt{m^2 + n^2}}$.

(4). If $1 = \left(\dfrac{\sin A}{\sin B}\right)^2 + (\cos A \,.\, \cos C)^2$; $\text{Sin}\, C = \dfrac{\tan A}{\tan B}$.

(5). If $\text{Cos}\, x = \dfrac{\cos A}{\sin C}$, and $\text{Cos}\,(90^0 - x) = \dfrac{\cos B}{\sin C}$;

$$(\cos A)^2 + (\cos B)^2 + (\cos C)^2 = 1.$$

7. Prove the following formulæ:

(1). $\operatorname{Tan} A + \cot A = 2 \operatorname{cosec} 2A$. (2). $\operatorname{Cot} A - \tan A = 2 \cot 2A$.

(3). $\operatorname{Tan} A = \dfrac{\sin 2A}{1 + \cos 2A}$. (4). $\dfrac{1 + \sin A}{1 - \sin A} = \dfrac{(1 + \tan \frac{1}{2} A)^2}{(1 - \tan \frac{1}{2} A)^2}$.

(5). $\operatorname{Cosec} 2A + \cot 2A = \cot A$. (6). $2 . \operatorname{Cosec} 2A = \sec A . \operatorname{cosec} A$.

(7). $\dfrac{\operatorname{Sin} A}{1 - \cos A} = \cot \frac{1}{2} A$. (8). $\dfrac{\operatorname{Versin} A}{\operatorname{versin}(180^\circ - A)} = (\tan \frac{1}{2} A)^2$.

(9). $\operatorname{Cot} 2A . \operatorname{cosec} 2A = \cot A . (\operatorname{cosec} A)^2 - \tan A . (\sec A)^2$.

(10). $\operatorname{Versin}(180^\circ - A) = 2 \operatorname{vers} \frac{1}{2}(180^\circ + A) . \operatorname{vers} \frac{1}{2}(180^\circ - A)$.

(11). $\operatorname{Sec} 2A = \dfrac{\cot A + \tan A}{\cot A - \tan A}$. (12). $(\operatorname{Tan} \frac{1}{2} A)^2 = \dfrac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A}$.

8. Prove that, *generally*,

(1). $\operatorname{Cos} 2A + \cos 2B = 2 . \cos(A + B) . \cos(A - B)$.

(2). $2 . \{(\sin A . \sin B)^2 + (\cos A . \cos B)^2\} = 1 + \cos 2A . \cos 2B$.

(3). $(\operatorname{Cos} \overline{A + B})^2 - (\sin A)^2 = \cos B . \cos(2A + B)$.

(4). $\dfrac{\operatorname{Sin}(A - B)}{\sin A . \sin B} + \dfrac{\sin(B - C)}{\sin B . \sin C} + \dfrac{\sin(C - A)}{\sin C . \sin A} = 0$.

(5). $\operatorname{Cos}(A + B) . \sin(A - B) + \cos(B + C) \sin(B - C)$

$+ \cos(C + D) . \sin(C - D) + \cos(D + A) . \sin(D - A) = 0$.

9. Prove that

(1). $\dfrac{1 + \sin A}{1 - \sin A} = \{\tan(45^\circ + \frac{1}{2} A)\}^2$.

(2). $\operatorname{Sec}(45^\circ + A) . \sec(45^\circ - A) = 2 \sec 2A$.

(3). $2 \sec A = \tan(45^\circ + \frac{1}{2} A) + \cot(45^\circ + \frac{1}{2} A)$.

(4). $\operatorname{Tan}(30^\circ + A) . \tan(30^\circ - A) = \dfrac{2 \cos 2A - 1}{2 \cos 2A + 1}$.

(5). $\operatorname{Sin}(60^\circ + A) - \sin(60^\circ - A) = \sin 2A$.

10. Find the numerical values of Sin 15°, Sin 9°, Cos 12°, Versin 15°, Tan 22°,12′, Sin 150°, Cos 135°, Sin 3°, Sin 75°, Sec 225°; and prove that

(1). $\text{Tan}\, 50^0 + \cot 50^0 = 2 \sec 10^0$.

(2). $2\cos\frac{A}{2^n} = \sqrt{2 + \sqrt{\{2 + \&c + \sqrt{(2 + 2\cos A)}\}}}$;

where n is an integer, and the symbol $\sqrt{}$ is repeated n times, —each time affecting all the quantities which follow it.

11. Determine A in the following equations:

(1). $\text{Sin}\, A = \sin 2A$.

(2). $\text{Tan}\, 2A = 3\tan A$.

(3). $\text{Tan}\, A = \text{cosec}\, 2A$.

(4). $\text{Cos}\, A = \tan A$.

(5). $(\text{Tan}\, 2A)^2 = 3\tan 2A$.

(6). $\text{Tan}\, A + 3\cot A = 4$.

(7). $\text{Tan}\, A . \tan 2A + \cot A = 2$.

(8). $2(\sin 3A)^2 + (\sin 6A)^2 = 2$.

(9). $\text{Tan}\, A + \cot 2A = \sin A . (1 + \tan A . \tan \frac{1}{2} A)$.

(10.) $\text{Cos}\, nA + \cos (n-2) A = \cos A$.

12. Determine x in the following equations:

(1). $\text{Sin}\,(a - x) = \cos (a + x)$.

(2). $\text{Sin}\,(x + a) + \cos (x + a) = \sin (x - a) + \cos (x - a)$.

(3). $\text{Sin}\, a + \sin (x - a) + \sin (2x + a) = \sin (x + a) + \sin (2x - a)$.

(4). $\text{Tan}\, a . \tan x = (\tan \overline{a + x})^2 - (\tan \overline{a - x})^2$; —to find $\cos x$.

(5). $m . \text{versin}\, x = n . \text{versin}\,(a - x)$.

(6). $\frac{m . \tan (a - x)}{(\cos x)^2} = \frac{n \tan x}{(\cos \overline{a - x})^2}$. $\left[\text{Tan}\,(a - 2x) = \frac{n - m}{n + m} . \tan a.\right]$

(7). $\text{Tan}\,(a + x) . \tan (a - x) = \frac{1 - 2\cos 2a}{1 + 2\cos 2a}$. $[x = 30^0.]$

(8). $n\,.\,(\sec x)^2.\ \tan(\alpha - x) = m\,.\,(\sec \overline{\alpha - x})^2\,.\ \tan x.$

$$\left[\mathrm{Tan}\,2x = \frac{m\sin 2\alpha}{m + n\cos 2\alpha}.\right]$$

(9). $\mathrm{Sin}\,x = \sin\alpha\,.\,\sin(\beta + x).$ $\left[\mathrm{Tan}\,x = \dfrac{\sin\alpha\,.\,\sin\beta}{1 - \sin\alpha\,.\,\cos\beta}.\right]$

(10). $\mathrm{Tan}\,x = \cos\alpha\,.\,\tan\beta.$ Shew that

$$\mathrm{Tan}(\beta - x) = \frac{(\tan\frac{1}{2}\alpha\,.\,\sin 2\beta)^2.}{1 + (\tan\frac{1}{2}\alpha)^2.\ \cos 2\beta}.$$

(11). $\dfrac{m}{n} = \dfrac{\sin\alpha}{\sin\beta}\cdot\dfrac{\cos(\beta + x)}{\cos(\alpha - x)}.$ Find $\mathrm{Tan}\,x$.

(12). If $2\sin(x - y) = 1$, and $\sin(x - y) = \cos(x + y)$;

then $x = 45^0$, and $y = 15^0$.

13. If $A + B + C = 90^0$;

(1). $\mathrm{Tan}\,A\,.\,\tan B + \tan A\,.\,\tan C + \tan B\,.\,\tan C = 1.$

(2). $\mathrm{Cot}\,A + \cot B + \cot C = \cot A\,.\,\cot B\,.\,\cot C.$

(3). $\mathrm{Tan}\,A + \tan B + \tan C = \tan A\,.\,\tan B\,.\,\tan C + \sec A\,.\,\sec B\,.\,\sec C.$

14. If $A + B + C = 180^0$;

(1). $4\sin A\,.\,\sin B\,.\,\sin C = \sin 2A + \sin 2B + \sin 2C.$

(2). $(\mathrm{Sin}\tfrac{1}{2}A)^2 + (\sin\tfrac{1}{2}B)^2 + (\sin\tfrac{1}{2}C)^2 + 2\sin\tfrac{1}{2}A\,.\,\sin\tfrac{1}{2}B\,.\,\sin\tfrac{1}{2}C = 1.$

15. Prove that,

(1). $\mathrm{Tan}^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = 45^0.$ (2). $\mathrm{Tan}^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{1}{3} = 45^0.$

(3). $\mathrm{Cot}^{-1}\tfrac{3}{4} + \cot^{-1}\tfrac{1}{7} = 135^0.$ (4). $\mathrm{Sin}^{-1}\dfrac{1}{\sqrt{5}} + \cot^{-1}3 = 45^0.$

(5). If $\theta = \tan^{-1}\dfrac{1}{\sqrt{3}}$, and $\phi = \tan^{-1}\dfrac{1}{\sqrt{15}}$;

$$\text{then}\quad \text{Sin}\,(\phi + \theta) = \sin 60^0 \,.\, \cos 36^0.$$

(6). If $\theta = \cos^{-1}\sqrt{\tfrac{2}{3}}$, and $\phi = \cos^{-1}\dfrac{\sqrt{3}+\sqrt{2}}{2\sqrt{2}}$; then $\phi + \theta = 60^0$.

(7). If $\text{Cot}\, x = n \,.\, \cot\,(a - x)$; then $2x = a - \sin^{-1}\left(\dfrac{n-1}{n+1} \,.\, \sin a\right)$.

(8). If $\text{Versin}^{-1}\dfrac{x}{a} - \text{versin}^{-1}\dfrac{bx}{a} = \text{versin}^{-1}\,(1 - b)$;

$$\text{then}\quad \frac{x}{a} = \pm\sqrt{\frac{2b}{1+b}}.$$

16. (1). If $m = \tan\theta - \sin\theta$, and $n = \tan\theta + \sin\theta$; find the relation between m and n.

(2). If $(x - a) \,.\, \cos\delta + c \,.\, \sin\,(l - \delta) = 0$

$(y - b) \,.\, \cos\delta + c \,.\, \cos\,(l - \delta) = 0$;

then $(x - a) \,.\, \sin l + (y - b) \cos l + c = 0$.

(3). If $a \,.\, (\sin\theta)^2 + a' \,.\, (\cos\theta)^2 = b$, $a' \,.\, (\sin\theta')^2 + a \,.\, (\cos\theta')^2 = b'$;

and $a \,.\, \tan\theta = a' \,.\, \tan\theta'$; then $\dfrac{1}{a} + \dfrac{1}{a'} = \dfrac{1}{b} + \dfrac{1}{b'}$.

(4). If $\text{Tan}\,\theta = (\tan\tfrac{1}{2}\phi)^3$, and $(\cos\phi)^2 = \dfrac{m^2 - 1}{3}$;

$$\text{then}\quad m = \frac{2}{\left\{(\cos\theta)^{\frac{2}{3}} + (\sin\theta)^{\frac{2}{3}}\right\}^{\frac{3}{2}}}.$$

(5). If $\text{Cos}\, v = \dfrac{\cos u - e}{1 - e\cos u}$; then $\text{Tan}\,\tfrac{1}{2} v = \sqrt{\dfrac{1+e}{1-e}} \,.\, \tan\tfrac{1}{2} u$.

17. In a RIGHT-ANGLED TRIANGLE ABC, C being the right angle;

(1). $\operatorname{Cos}(2A - B) = \frac{a}{c^3} \cdot (3c^2 - 4a^2)$.

(2). Area = (semi-perimeter) × (semi-perimeter − hypothenuse).

(3). The perimeter being given, and the perpendicular from C on c, to find the three sides.

(4). A person observes the elevation of a tower to be 60°, on receding from it 100 yards he finds the elevation to be 30°; required the height of the tower.

(5). A cylindrical tower of given diameter terminates in a conical spire. At a measured distance from the foot of the tower the angles of elevation of the tops of the spire and the tower are observed. Find the heights of the tower and the spire, (1) when the measured distance is horizontal, and (2) when it is inclined at a given angle to the horizon.

(6). To a person standing 288 yards from the foot of a tower, its elevation is 53°, 8′; determine the height of the tower:

$$L \tan 53^0, 8' = 10{\cdot}124 ; \quad 1_{10} 288 = 2{\cdot}459.$$

(7). The hypothenuse is 7854·3, and one angle is 37°, 18′; find the logarithms of the opposite side.

$$L \operatorname{cosec} 37^0, 18' = 10{\cdot}2175357 ; \quad 1_{10} 7{\cdot}8543 = {\cdot}8951075.$$

OBLIQUE-ANGLED TRIANGLES.

18. General Properties.

(1). $\operatorname{Tan} B = \frac{b \cdot \sin C}{a - b \cdot \cos C}$.

(2). $\operatorname{Cos} \frac{1}{2}(A-B) = \frac{a+b}{c} \cdot \sin \frac{1}{2} C$; $\quad \operatorname{Sin}(A-B) = \frac{a^2-b^2}{c^2} \cdot \sin C$.

(3). $\frac{1}{2} \cdot (a^2 + b^2 + c^2) = ab \cdot \cos C + ac \cdot \cos B + bc \cdot \cos A$.

(4). $\frac{1}{2} . (a+b+c) = \frac{1}{2}(a+b) . \cos C + \frac{1}{2}(a+c) \cos B + \frac{1}{2}(b+c) . \cos A.$

(5). $c^2 = (a+b)^2 . (\sin \frac{1}{2} C)^2 + (a-b)^2 . (\cos \frac{1}{2} C)^2.$

(6). $2a . \cot \frac{1}{2} A = (b+c-a) . (\cot \frac{1}{2} B + \cot \frac{1}{2} C).$

(7). $\frac{a}{b} . (\sin \frac{1}{2} B)^2 = \frac{S-a}{S-b} . (\sin \frac{1}{2} A)^2.$

(8). The length of the line from C which bisects c is

$$\frac{1}{2} . \{2(a^2+b^2) - c^2\}^{\frac{1}{2}}.$$

(9). Area $= \frac{1}{2}(a^2 - b^2) . \frac{\sin A . \sin B}{\sin (A-B)}.$

(10). $\mathrm{l}_\epsilon \frac{a}{b} = (\cos 2A - \cos 2B) + \frac{1}{2} . (\cos 4A - \cos 4B)$

$$+ \frac{1}{3} . (\cos 6A - \cos 6B) + \ldots$$

(11). $\mathrm{l}_\epsilon \frac{a}{c} = \frac{b}{a} . \cos C + \frac{b^2}{2a^2} . \cos 2C + \frac{b^3}{3a^3} \cos 3C + \ldots.$

(12). $B'' = \frac{b}{a} . \frac{\sin C}{\sin 1''} + \frac{b^2}{a^2} . \frac{\sin 2C}{\sin 2''} + \frac{b^3}{a^3} . \frac{\sin 3C}{\sin 3''} + \ldots.$

19. (1). The sides of a triangle are 13, 12, 5; find its area.

(2). Given the perimeter, area, and an angle, find the side opposite to the given angle.

(3). Given B, a, and area, to determine the triangle.

(4). Given the vertical angle, the perpendicular on the base, and the rectangle under the segments of the base, to determine the triangle.

(5). The length of a road, in which the ascent is 1 foot in five, from the foot of a hill to the top is a mile and two-thirds. What will be the length of a zigzag road in which the ascent is 1 foot in 12?

(6). The area of a triangle is $\frac{4}{3}$ of the triangle whose sides are the lines joining the angles of the first triangle with the bisections of the sides respectively opposite.

(7). If in the three edges of a cube which meet in a point O, points A, B, C, be taken which are distant a, b, c, from O, the area of the triangle formed by joining A, B, C, is $\frac{1}{2}(a^2b^2 + b^2c^2 + a^2c^2)^{\frac{1}{2}}$.

(8). The sides of a triangle are as $n-1$, n, $n+1$, where n is very large; find the angles, and determine the difference of each from 60^0.

(9). An object six feet high, standing at the top of a tower, subtends an angle $\text{Tan}^{-1}\cdot 015$ at a station which is 100 yards from the base. Find the height of the tower.

(10). Two objects, A and B, were observed to be at the same instant in a line inclined at an angle 15^0 to the east of a ship's course which was at the time due north. The ship's course was then altered, and after sailing 5 miles in a N. W. direction, the same objects were observed to bear E. and N. E. respectively. Required the distance of A from B.

(11). A vessel observed another a^0 from the north, sailing in a direction parallel to its own. In p hours its bearing was β^0, and in q hours afterwards γ^0 from the north. To what point of the compass were the vessels sailing?

(12). The shadows of two vertical walls, which are at right angles to each other, and are a and a_1 feet in height, are observed, when the Sun is due south, to be b and b_1 feet in breadth. Shew that if a be the Sun's altitude above the horizon, and β be the inclination of the first wall to the meridian;

$$\text{Cot}\, a = \sqrt{\left(\frac{b^2}{a^2} + \frac{b_1{}^2}{a^2}\right)}. \qquad \text{Cot}\, \beta = \frac{a b_1}{b a_1}.$$

(13). A boy was flying a kite at noon when the wind was blowing α^0 from the south, and the angular distance of the kite's shadow from the north was β^0. The wind suddenly changed to α'^0 from the south, and the shadow to β'^0 from the north, and the kite was raised as much above 45^0 as it had before been below that elevation. Shew that θ^0 being the angular elevation of the Sun, and $45^0 - \phi^0$ that of the kite at first,

$$(\text{Tan}\,\theta^0)^2 = \frac{\sin\beta \, . \, \sin\beta'}{\sin(\alpha-\beta) \, . \, \sin(\alpha'-\beta')},$$

$$(\text{Tan}\,\overline{45^0-\phi^0})^2 = \frac{\sin(\alpha-\beta) \, . \, \sin\beta'}{\sin(\alpha'-\beta') \, . \, \sin\beta}.$$

(14). A privateer observes the direction in which a trader leaves a harbour at a known distance. Having given the rates of sailing of the two vessels, shew how to compute the angle at which the privateer must pursue the trader, so that she may just have prepared for action at the time she comes within gun-shot.

(15). There are two towns lying N. and S. of each other, and distant $2\frac{1}{4}$ miles. Their angles of depression as observed from a balloon are 45^0 and 60^0, and after the balloon has proceeded 6 miles in a S.E. direction, the angles of depression of each are $\frac{1}{2}$ of what they were before. Shew that the height of the balloon was 3 miles very nearly.

(16). The sides a, b, c of a triangle are 4, 5, 6, find the angle B; having given

$l_{10}2 = \cdot 3010299$. $L\cos 27^0, 53' = 9\cdot 9464040$.

$l_{10}5 = \cdot 6989700$. $L\cos 27^0, 54 = 9\cdot 9463371$.

(17). $l_{10}3 = \cdot 4771213$, $l_{10}\tan 57^0, 19', 11'' = \cdot 1928032$. Shew that if one angle be 60^0, and the two sides containing it are as 19 to 1, the other two angles are $117^0, 19', 11''$, and $2^0, 40', 49''$.

(18). Two sides of a triangle are as 9 to 7, and the included angle is $64^0, 12'$; determine the degrees, minutes, and seconds in the other angles. Given

$$l_{10}2 = 3010300, \qquad L \tan 57^0, 54' = 10{\cdot}2025255.$$

$$L \tan 11^0, 16' = 9{\cdot}2993216. \qquad L \tan 11^0, 17' = 9{\cdot}2999804.$$

20. (1). The radii of the circles described within and about a triangle whose sides are a, b, c being r and R, prove that

$$2Rr = \frac{abc}{a+b+c}; \quad \text{and } r = 4R \,.\, \sin \tfrac{1}{2} A \,.\, \sin \tfrac{1}{2} B \,.\, \sin \tfrac{1}{2} C.$$

(2). The product of the radii of the four circles which touch three lines which intersect each other = the square of the area of the triangle which the three lines include.

(3). Three circles whose radii are a, b, c, touch each other externally; prove that the tangents at the points of contact meet in a point, whose distance from any one of them is,

$$= \left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.$$

(4). The areas of all triangles described about a given circle are as their perimeters.

(5). If a, a' be homologous sides of similar triangles described about and within a circle,

$$a' = 4a \,.\, \sin \tfrac{1}{2} A \,.\, \sin \tfrac{1}{2} B \,.\, \sin \tfrac{1}{2} C.$$

21. (1). In a regular polygon of n sides, of which a side is $2a$,

$$R + r = a \,.\, \cot \frac{\pi}{2n}.$$

(2). The square of a side of a pentagon inscribed in a circle = square of a side of the inscribed hexagon, together with the square of the side of the inscribed decagon.

(3). The area of a circle and that of a regular polygon inscribed in it being given, find the number of the sides of the polygon, and the angles they make one with another.

22. Adapt to logarithmic computation;

(1). $\sqrt{a+b} + \sqrt{a-b}$.

(2). $\dfrac{\sqrt{a+b}}{a-b} + \dfrac{\sqrt{a-b}}{a+b}$.

In the following questions θ is required.

(3). $\text{Sin}\,(\theta + a) = m\,.\,\sin\theta$.

(4). $\text{Cos}\,(a + \theta) = \cos a\,.\,\sin\theta + \sin\beta$.

(5). $\text{Cos}\, m\theta + \cos n\theta + \cos p\theta = 1$.

(6). $\text{Tan}\,\theta = \dfrac{na\,.\cos A - b\,.\sin B}{ma\,.\cos A + c\,.\sin C}$; where A, B, C are the angles of a triangle.

(7). $\text{Sin}\theta\,.\sqrt{\{1 + (\tan A\,.\tan B)^2\}} + \cos\theta\,.\sqrt{\{1 - (\tan A\,.\tan B)^2\}}$
$= \tan A + \tan B$.

23. Prove that,

(1). $\frac{2}{3}\,.\,(8\sin\frac{1}{4}\theta - \sin\frac{1}{2}\theta) = \theta$; nearly.

(2). $\text{Versin}\,\theta = -\frac{1}{2}\,.\,\{\epsilon^{\frac{1}{2}\theta\sqrt{-1}} - \epsilon^{-\frac{1}{2}\theta\sqrt{-1}}\}$.

(3). If from an arc AB of a circle of radius r whose centre is O, AC be cut off $= r\,.\sin AOB$; then sector COB = segment ACB.

24. Solve the equations;

(1). $x^6 + 1 = 0$.

(2). $x^3 - 6x - 4 = 0$. $[x = -2, \text{ or } 1 \pm \sqrt{3}.]$

(3). $x^3 - 147x - 343 = 0$. [One root is 13·1556964; given $\cos 20^0 = \cdot 9396926$.]

---

25. LOGARITHMS.

(1). From $l_{10}2$, and $l_{10}3$, find $l_{10}72$.

(2). From $l_{10}14$, and $l_{10}16$, find $l_{10}2$, $l_{10}5$, $l_{10}7$, $l_{10}\frac{1}{1960}$.

(3). From $l_{10}2$, and $l_{10}67$, find $L \tan 15^0$.

(4). $l_e\{(1+N)^{\frac{1+N}{2}} . (1-N)^{\frac{1-N}{2}} = \frac{N^2}{1.2} + \frac{N^4}{3.4} + \frac{N^6}{5.6} + \ldots\}$.

2

THE ELEMENTS OF SPHERICAL TRIGONOMETRY.

BIBLIOTHECA BODLEIANA

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SECOND EDITION.

CAMBRIDGE:
PRINTED BY JOHN W. PARKER, UNIVERSITY PRINTER.
T. STEVENSON, CAMBRIDGE;
A. H. BAILY & Co., LONDON.

M.DCCC.XXXVII.

SPHERICAL TRIGONOMETRY.

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1—30, 35—39, 41—43, 46—56, 62—64.

SPHERICAL TRIGONOMETRY.

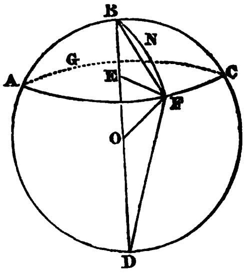
CHAPTER I.

ON CERTAIN PROPERTIES OF SPHERICAL TRIANGLES.

1. Def. A *Sphere* is a solid bounded by a surface of which every point is equally distant from a fixed point, called the *centre* of the sphere.

Def. The line joining the centre with any point in the surface is called the *radius* of the sphere.

2. *Every section of a sphere made by a plane is a circle.*



Let $ABCD$ be a sphere of which the center is O; $AFCG$ the curve in which a plane cutting the sphere intersects its surface; OE a perpendicular from O upon the cutting plane.

Join E with F any point in $AFCG$, and join FO. Then since OE is perpendicular to the cutting plane, it is perpendicular to EF, a line in that plane;

$$\therefore\ EF = \sqrt{(OF^2 - OE^2)}, \text{ a constant quantity.}$$

Now E is a *fixed* point in the cutting plane, and F is *any* point in the curve AFC. Therefore AFC is a circle whose center is E and radius is EF. Euclid, I. Def. 15.

Cor. If the cutting plane pass through the centre of the sphere, EO vanishes, and EF becomes equal to OF, the radius of the sphere.

3. Def. The circle in which a sphere is cut by a plane is called a *great*, or a *small*, circle according as the cutting plane passes, or does not pass, through the centre of the sphere.

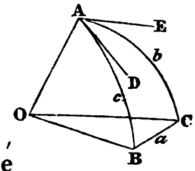
Cor. Since the radius of every great circle is equal to the radius of the sphere, Art. 2, all great circles of the sphere must be equal.

Note. Unless the contrary be expressly mentioned, when hereafter we speak of an arc of a sphere, an arc of a *great* circle is meant.

4. Def. Spherical Trigonometry investigates the relations subsisting between the angles of the plane faces which form a solid angle, and the angles at which the plane faces themselves are inclined to one another.

For the sake of convenience the angular point is made the centre of a sphere, which being cut by the plane faces containing the solid angle, presents on its surface a figure whose sides are arcs of great circles.

Let ABC be a triangle of this kind, whose sides AB, BC, CA, are formed by the intersection of the planes AOB, BOC, COA, with the surface of a sphere of which O is the centre. The angle of any face, as AOB, is $\frac{\text{arc}\, AB}{AO}$, Pl. Trig. Chap. VI.; and the angle contained between any two faces, as BAO and CAO, is the angle contained between AD and AE, which are lines drawn in the planes BAO and CAO at right angles to their intersection AO. Eucl. XI. Def. 6.



The lines AD, AE, being at right angles to the radius OA, and lying in the planes of the arcs AB and AC, are tangents to those arcs.

5. After certain properties of spherical triangles have been proved, it will not, in pursuing further investigations, be requisite to represent in our figures the centre of the sphere on which the triangles are described.

Note. It must, however, never be forgotten, that when the words "the angle *BAC*" or "the angle *A*" occur, the angle meant is that of the inclination of the planes passing through *O*, the centre of the sphere, and the arcs of *AB* and *AC*,—and that this is *the angle contained between two lines drawn from any point in* AO *at right angles to it, and respectively lying in the planes* AOB *and* AOC. Also, whenever the expression "the sine of *AB*" occurs, the sine is meant of the angle which the arc *AB* subtends at the centre of the circle of which it is a portion.—If this circle be a great circle, its centre is also the centre of the sphere.

6. In the following pages the angle *BAC* will commonly be indicated by the letter *A*, and the angle subtended by *BC*, the side of the triangle opposite to $\angle BAC$, will be indicated by *a*. The other parts of the triangle *BAC* will be represented in like manner.

7. Def. If *OE*, which is perpendicular to the plane *AFC*, (Fig. Art. 2.) be produced both ways to meet the surface of the sphere in *B* and *D*, these points are respectively called *the nearer* and *the more remote poles* of the circle *AFC*, and the straight line *BOD* is called *the axis of the circle AFC*.

8. *The pole of a circle is equally distant from every point in its circumference.* (Fig. Art. 2.)

Join *B* with *F* any point in *AFCG*. Then, *BEF* being a right angle,

$$BF^2 = BE^2 + EF^2$$

$$= BE^2 + \overline{OF^2 - OE^2}; \text{ a constant quantity.}$$

And *F* being *any* point in *AFCG*, *B* is equally distant from every point in the circumference of that circle.

Similarly, $DF^2 = DE^2 + EF^2$

$$= DE^2 + \overline{OF^2 - OE^2}; \text{ a constant quantity.}$$

Hence *D* is equally distant from every point in the circumference of *AFCG*.

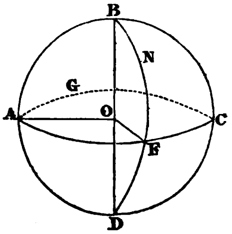
Again; let *BF* be an arc of a great circle passing through *B* and *F*. Then since *BF* is constant, and in equal circles, equal circumferences are subtended by equal straight lines, Euclid III. xxix, the arc *BNF* is constant,—and also, because the radii of all great circles are equal, the angle *BOF* subtended at the centre of the sphere, is constant.

Hence it appears that every point in the circumference of a circle of a sphere is equally distant from the pole of the circle; whether the distance be estimated by the length of a straight line joining the point with the pole,—or by the arc of a great circle connecting the same points,—or by the angle which this arc subtends at the centre of the sphere.

9. Since *BO* is at right angles to the plane *AFC*, every plane through *BO* is at right angles to that plane. Hence, *the angle between any circle whatever and a great circle passing through its pole, is a right angle.*

10. If *AFCG* become a great circle, *OE* vanishes, *E* coincides with *O*, and *BOF* becomes a right angle.

Now since *BO* is perpendicular to the plane *AOF*, the angle *BOF* is a right angle, and *BF* is a quadrant.



Also, since *BOF* is a plane passing through *BC*, which is a line perpendicular to the plane *AOF*, *BOF* is at right angles to *AOF*, and therefore the angle *BFA*—which, Art. 4, is the inclination of the two planes to one another,—is a right angle.

Hence it appears, that if a quadrant (*FB*) be drawn at right angles to a great circle *AFC* from any point *F* in it; the extremity of the quadrant is the pole of the circle.

Cor. If *B* be the pole of *AFC*, *AO* and *FO* are at right angles to *BO*, and *AOF* is the inclination of *AOB* and *FOB*;

$$\therefore \text{ Art. 5, } \angle AF = \angle ABF.$$

Def. Great circles which pass through the pole of a great circle are called *secondaries* to that circle.

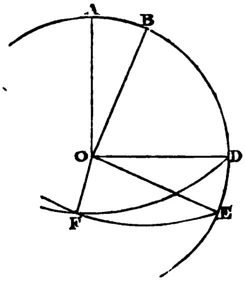
11. *If from a point on the surface of a sphere there can be drawn two arcs (which are not parts of the same circle) at right angles to a given circle, that point is the pole of the circle.* (Fig. Art. 10.)

Let *B* be the point, *BA* and *BF* arcs through *A* and *F*, which are points in the circle *AFC*; and let *BA* and *BF* be at right angles to *AF*.

Then since *BA* and *BF* are at right angles to *AF*, and are not parts of the same circle, the planes *BAO* and *BFO* must intersect, and their intersection *BO* is at right angles to *AOF*, Eucl. xi. 19;—therefore *B* is the pole of *AFC*.

Cor. Hence also it appears, that the intersection of two arcs drawn at right angles to a given circle through any two points in it, is the pole of that circle.

12. *The angle subtended at the centre of the sphere by the arc which joins the poles of two great circles, is the inclination of the planes of the circles.*



Let the given circles be *FD* and *FE* intersecting in *F*, *A* and *B* their respective poles, and *ABD* the circle through *A* and *B*.

Now *AO* is perpendicular to *OF*,—a line in the plane *DOF*,

And *BO* is perpendicular to *OF*,—a line in the plane *EOF*;

∴ *OF* is perpendicular to the plane *AOB*, and therefore to *OD* and *OE*, which are lines in that plane;

∴ *DOE* is the angle of inclination of the planes *FOD*, *FOE*.

$$\begin{aligned} \text{And } AOB &= AOD - BOD = 90^{\circ} - BOD \\ &= BOE - BOD \\ &= DOE. \end{aligned}$$

Cor. Hence also it appears that $\angle AB = \angle DE = \angle DFE$. Arts. 5, 4.

13. *Two great circles are bisected by their intersections.* (Fig. Art. 10.)

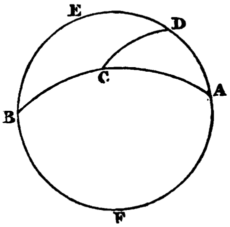
Let *BAD* and *BFD* be arcs of two great circles intersecting each other in *B* and *D*. Join *BD*, and let *O* be the center of the sphere.

Then since *BAD* is a great circle, *O* is a point in its plane; similarly, *O* is a point in the plane of *BFD*; therefore *O* lies in the intersection of these planes, that is, in the line joining *B* and *D*. Therefore *BAD* and *BFD* are semicircles, having *BOD* for their common diameter, and the two great circles are bisected by the points *B* and *D*.

14. *Any side of a spherical triangle is less than a semicircle, and any angle is less than two right angles.*

Since Euclid takes two right angles as the limit of a plane angle, this is also his limit for the angle of any plane face of a solid angle*. Hence in a spherical triangle no side can be equal to a semicircle.

Thus, if *ACB* intersect *AEBF* in the points *A* and *B*, and *CD* be any other arc, then the triangle connecting the points *B*, *C*, *D* is *not* the figure formed by *BC*, *CD*, *DAFB*, (of which the side *DAFB* is greater than a semicircle), but the figure formed by *BC*, *CD*, *DEB*.



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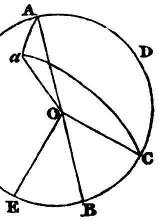
[*] If a face could have an angle greater than two right angles, Euclid's Proposition, xi. 20., "If a solid angle be contained by three plane angles, any two of them are together greater than the third," would not be true in all cases.

For let *ACB* be a circle whose diameter is *AOB*, and let *COA* revolve about *OC*, until *A* comes into the position *a*. Now if *AOa* be less than *BOC*, it is evident that if the three plane angles containing the solid angle whose edges are *AO*, *aO*, *CO* be those subtended by the arcs *Aa*, *aC*, *CEA*, two of them, namely those subtended by *Aa* and *aC*, are together not equal to the third, which is subtended by *CEA*.



If, following Euclid's construction in xi. 20., we made *AOE*=*AOC*, and joined *AE*, Euclid's proof would fail from *EA* not cutting the edge *OC*.

The faces of the solid angle whose edges are *AO*, *aO*, *CO*, are the sectors subtended by *Aa*, *aC*, *CDA*.

If a side, as ADB, become a semicircle, then the arcs which join A and B with any other point C, are portions of the semicircle BCA, and the arcs joining the points A, B, C cease to form a triangle. Wherefore the *side* of a triangle is less than a semicircle.

Cor. 1. Hence it follows that an *angle* of a triangle must be less than two right angles.

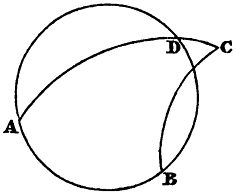
For if possible let $BFADC$ be a triangle having BCD greater than two right angles. Produce BC to A; then BFA is a semicircle, and $BFAD$ is greater than a semicircle; which is impossible.

If BCD become equal to two right angles, BC and CD become parts of the same circle, and the figure ceases to be a triangle.

Wherefore the *angle* of a triangle is less than two right angles.

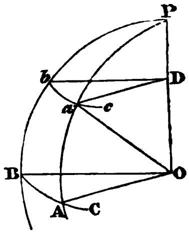
Cor. 2. Hence it follows, that if the great circle be completed of which any side AB of a triangle ABC forms a part, the triangle lies within it.

For if possible, let C lie without the circle, and let CA cut it in D; then since DA is a semicircle, Art. 13, CDA is greater than a semicircle; which is impossible.



15. *If* P *be the pole of a great circle* BAC *and of a small circle* bac *which are cut by the great circles* PaA *and* PbB, *then* $\frac{\text{arc ab}}{\text{arc AB}} = \sin \text{Pa}$.

Let OP, which is at right angles to the planes of both circles, cut the plane of the small circle in D. Join AO, BO, aD, bD. Since these lines lie in planes which are perpendicular to OP, each of them is perpendicular to OP. Join aO.



Then aD is perpendicular to PO, and is therefore parallel to AO, which lies in the same plane with aD, and is perpendicular to PO.

Similarly, bD is parallel to BO;

$\therefore\ \angle aDb = \angle AOB.$ Eucl. XI. 10.

that is, $\dfrac{\text{arc}\,ab}{aD} = \dfrac{\text{arc}\,AB}{AO}$;

$$\therefore\ \frac{\text{arc}\,ab}{\text{arc}\,AB} = \frac{aD}{AO} = \frac{aD}{aO} = \sin aOP, \quad \text{or } \sin Pa.$$

Cor. 1. Since $POA = 90^0$, $\sin POa = \cos aOA$;

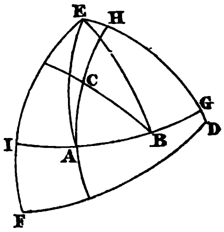
$$\therefore\ \frac{\text{arc}\,ab}{\text{arc}\,AB} = \cos aOA, \quad \text{or } \cos aA.$$

Cor. 2. If PA be not a quadrant, it may in like manner be proved that

$$\frac{\text{arc}\,ab}{\text{arc}\,AB} = \frac{\sin Pa}{\sin PA}.$$

16. Def. If the angular points A, B, C of a spherical triangle ABC, Fig. Art. 17, be made the poles of the three great circles DE, EF, FD respectively, the triangle ABC is called with respect to DEF the *primitive* triangle, and DEF is called with respect to ABC the *polar* triangle.

17. *The angular points of the polar triangle are the poles of the sides of the primitive triangle.*



Join AE and BE, by arcs of two great circles.

Then, since A is the pole of ED, $AE = \dfrac{\pi}{2}$, Art. 10.

And, since B is the pole of EF, $BE = \dfrac{\pi}{2}$.

Therefore the great circle of which E is the pole passes through A and B, or E is the pole of AB. Similarly D and F are the poles of AC and BC respectively.

18. The *polar* triangle is called also the *supplemental* triangle, from the following property it possesses.

The sides and angles of the polar triangle are the supplements of the angles and sides respectively of the primitive triangle. (Fig. Art. 17.)

$$\begin{aligned} \text{For } \angle A &= HG \quad \text{Art. 10. Cor.} \\ &= EG - EH \\ &= EG - (ED - HD) \\ &= EG + DH - ED. \end{aligned}$$

But EG and DH are quadrants, or each subtends an angle $\frac{\pi}{2}$;

$$\therefore \angle A = \pi - ED.$$

Similarly, $\angle B = \pi - EF$, and $\angle C = \pi - FD$.

Again, $AB = AG - BG = AG - (IG - BI)$

$$= AG + BI - IG$$

$$= \frac{\pi}{2} + \frac{\pi}{2} - E. \quad \text{Since } E = IG. \quad \text{Art. 10. Cor.}$$

$$= \pi - E.$$

Similarly, $BC = \pi - F$, and $AC = \pi - D$.

19. *If a general equation be established between the sides and the angles of a spherical triangle, a true result is obtained if in the equation the supplements of the sides and of the angles respectively be written for the angles and sides which enter into the equation.*

For if the equation be proved for any triangle whatever, we may substitute in it the sides a', b', c', and the angles A', B', C', of the *polar* triangle in the place of the sides a, b, c, and the angles A, B, C, of the *primitive* triangle. And in the equation as it then stands, putting for the sides and angles of the polar triangle their equivalents drawn from the primitive triangle, viz.

$$\begin{aligned} \pi - A &= a', & \pi - a &= A', \\ \pi - B &= b', & \pi - b &= B', \\ \pi - C &= c', & \pi - c &= C', \end{aligned}$$

we obtain a true result, which differs from the original equation, in having the supplements of the sides and of the angles respectively written for the angles and sides of the triangle.

Ex. If A, B, C be the angles, and a, b, c be the sides of any triangle, it will hereafter be proved that,

$$\cos A = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}.$$

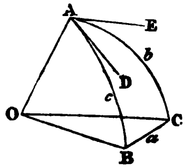
Therefore $$\cos A' = \frac{\cos a' - \cos b' \,.\, \cos c'}{\sin b' \,.\, \sin c'}.$$

Now $A' = \pi - a$, $a' = \pi - A$, $b' = \pi - B$, $c' = \pi - C$;

$$\therefore \quad \cos(\pi - a) = \frac{\cos(\pi - A) - \cos(\pi - B) \,.\, \cos(\pi - C)}{\sin(\pi - B) \,.\, \sin(\pi - C)}.$$

$$\therefore \quad \cos a = \frac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C}.$$

20. ***Any two sides of a spherical triangle are together greater than the third, and the sum of the three sides is less than the circumference of a great circle.***



For, Eucl. xi. 20., any two of the angles AOB, BOC, COA, $\left(\text{or } \frac{\text{arc } AB}{AO}, \frac{\text{arc } BC}{AO}, \frac{\text{arc } CA}{AO}\right)$, which form the solid angle at O, are together greater than the third. Therefore any two of the arcs AB, BC, CA are together greater than the third.

Also, Eucl. XI. 21., the three angles forming the solid angle at O are less than four right angles;

$$\therefore \quad \frac{\text{arc } AB}{AO} + \frac{\text{arc } BC}{AO} + \frac{\text{arc } CA}{AO} < 2\pi;$$

$$\therefore \quad \text{arc } AB + \text{arc } BC + \text{arc } CA < 2\pi . AO,$$ which is the circumference of a circle whose radius is AO.

Cor. 1. Since the sum of the plane angles which contain *any* solid angle is less than four right angles, Eucl. XI. 21., it follows from the same mode of proof, that the sum of the sides of *any* polygonal figure described on a sphere, whose sides are arcs of great circles, is less than the circumference of a great circle.

The polygonal figure, however, must be such that its area is contained on the surface of a hemisphere. For in the proof of the proposition of Euclid to which we have referred, it is supposed that *all* the plane faces which contain the solid angle may be cut by *one* plane,—which cannot be the case unless all the edges of the solid angle lie within the same hemisphere.

Cor. 2. Also let $ABCDE$ be a five-sided figure described on a sphere, and let it be divided into triangles by the arcs AC, AD. (The Student will easily draw the figure.)

Then $AB + BC > AC$;

$$\therefore \quad AB + BC + CD > AC + CD, \quad \text{and } AC + CD > AD;$$

$$\therefore \quad AB + BC + CD > AD;$$

$$\therefore \quad AB + BC + CD + DE > AD + DE > AE.$$

And the same method of proof being applicable to a polygon of any number of sides, it follows that the sum of all the sides but one of a spherical polygon is greater than the remaining side.

Cor. 3. If a and b be two sides of a spherical triangle, since each is less than π,

$$\therefore \quad a+b<2\pi, \qquad \therefore \quad \tfrac{1}{2}(a+b)<\pi.$$

$$\text{And} \quad a-b<\pi, \qquad \therefore \quad \tfrac{1}{2}(a-b)<\frac{\pi}{2}.$$

Also, since any angle of a triangle is less than π;

$$\therefore \quad A+B<2\pi, \qquad \therefore \quad \tfrac{1}{2}(A+B)<\pi.$$

$$\text{And} \quad A-B<\pi, \qquad \therefore \quad \tfrac{1}{2}(A-B)<\frac{\pi}{2}.$$

Cor 4. a', b', c', being the sides of the polar triangle,

$$a'+b'>c';$$

$$\therefore \quad (\pi-A)+(\pi-B)>(\pi-C);$$

$$\therefore \quad \pi>A+B-C.$$

$$\text{Or,} \quad A+B-C<\pi.$$

21. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*

Let A, B, C be the angles, and a, b, c the sides of a triangle; A', B', C' and a', b', c', the angles and sides of its polar triangle.

$$\text{Now} \quad 2\pi>a'+b'+c'. \quad \text{Art. 20.}$$

$$>(\pi-A)+(\pi-B)+(\pi-C)$$

$$>3\pi-(A+B+C);$$

$$\therefore \quad A+B+C>\pi.$$

Again, since each of the angles A, B, C is less than π,

$$\therefore \quad A+B+C<3\pi.$$

22. *The angles at the base of an isosceles triangle are equal to each other.* (Fig. Art. 28.)

Suppose ABC to be an isosceles triangle, having $AB = AC$, and therefore $\angle AOB = \angle AOC$.

From D, any point in OA, draw DG perpendicular to the plane BOC, and therefore at right angles to every line it meets in that plane; and from G draw GE and GF perpendicular to OB and OC; join ED, FD, OG.

$$\begin{aligned} \text{Then}\quad OE^2 &= OG^2 - GE^2 \\ &= (OD^2 - DG^2) - (ED^2 - DG^2) \\ &= OD^2 - ED^2; \end{aligned}$$

$\therefore$ DE is perpendicular to OE, and $\therefore$ $\angle DEG$ = inclination of the planes BOC and BOA, $= \angle B$.

Similarly DF is perpendicular to OC, and $\angle DFG = \angle C$.

$$\text{Now } DE = OD \,.\, \sin AOB = OD \,.\, \sin AOC = DF.$$

$$\text{And}\quad EG^2 = DE^2 - DG^2 = DF^2 - DG^2 = FG^2.$$

Hence, since GE, $ED = GF$, FD, and GD is common,

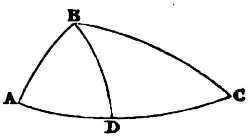
$$\therefore \quad \angle DEG = \angle DFG, \quad \text{or } \angle B = \angle C.$$

23. Conversely, if $\angle B = \angle C$, or $\angle DEG = \angle DFG$, it may be shewn from the same figure that $AB = AC$;—that is, the angles at the base being equal, the sides opposite to them are equal.

Cor. Hence also it follows that every equilateral triangle is also equiangular; and conversely, that every equiangular triangle is also equilateral.

24. *Of the two sides which are opposite to two unequal angles in a triangle, that is the greater which is opposite to the angle which is the greater.*

Let $\angle ABC$ be greater than $\angle BAC$. Then $AC > BC$.



Make $\angle DBA = \angle DAB$; $\therefore DA = DB$.

But $CD + DB > BC$. Art. 20.

$$\text{And} \quad AC = CD + DA = CD + DB;$$

$$\therefore \quad AC > BC.$$

25. Conversely, it may easily be shewn that of two angles in a triangle opposite to unequal sides, that is the greater which is opposite to the side which is the greater.

Cor. Hence $A - B$ and $a - b$ have the same sign.

26. Article 24 has been proved after the manner of Euclid I. 19. The following propositions may be enunciated for spherical triangles, in the terms used for plane triangles, and may be proved in nearly the same words. Euclid I. Props. 4. 8. 24. 25. 26. &c.

27. Before we proceed to prove certain formulæ which are useful for the solution of triangles, we will recapitulate some properties of triangles to which we shall have occasion to refer.

Arts. 14.	1. A side must be less than a semicircle.
14.	2. An angle must be less than two right angles.
20.	3. Any two sides are together greater than the third.
24.	4. The greater side is opposite to the greater angle; and conversely.
25.	5. $A - B$ and $a - b$ are of the same sign.
20.	6. $a + b + c < 2\pi$.
21.	7. $A + B + C > \pi$, and $< 3\pi$.
20. Cor. 3.	8. $a + b < 2\pi$, $a - b < \pi$.
20. Cor. 3.	9. $A + B < 2\pi$, $A - B < \pi$.
20. Cor. 4.	10. $A + B - C < \pi$.

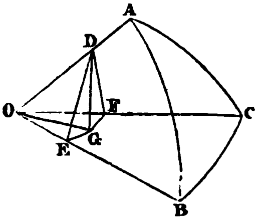
CHAPTER II.

FORMULÆ CONNECTING THE SIDES AND ANGLES OF A SPHERICAL TRIANGLE.

28. *In any spherical triangle, the sines of the angles are proportional to the sines of the sides respectively opposite: or,* ABC *being the triangle,*

$$\frac{\sin \mathrm{A}}{\sin \mathrm{BC}} = \frac{\sin \mathrm{B}}{\sin \mathrm{AC}} = \frac{\sin \mathrm{C}}{\sin \mathrm{AB}}.$$

Let O be the centre of the sphere, D any point in OA. Draw DG perpendicular to the plane BOC, DE and DF in the planes AOB and AOC perpendicular to OB and OC. Join EG, FG, OG.



Then, since DG is perpendicular to the plane BOC, and therefore to EG, a line in that plane,

$$EG^2 = ED^2 - DG^2 = (OD^2 - OE^2) - (OD^2 - OG^2) = OG^2 - OE^2;$$

$$\therefore\ EG \text{ is at right angles to } OE.$$

And ED, EG being each at right angles to OB, the intersection of the planes AOB and BOC, the angle DEG is the inclination ($\angle B$) of those planes.

$$\text{Similarly, } \angle DFG = \angle C.$$

Now $DE \cdot \sin B = DG = DF \cdot \sin C$;

$$\therefore \frac{\sin B}{\sin C} = \frac{DF}{DE} = \frac{\frac{DF}{OD}}{\frac{DE}{OD}} = \frac{\sin AOC}{\sin AOB} = \frac{\sin AC}{\sin AB};$$

$$\therefore \frac{\sin B}{\sin AC} = \frac{\sin C}{\sin AB}.$$

And by letting fall a perpendicular from any point in OC upon the plane AOB, it might in like manner be shewn that

$$\frac{\sin A}{\sin BC} = \frac{\sin B}{\sin AC}.$$

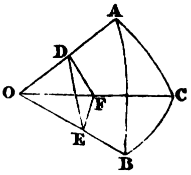
Wherefore,

$$\frac{\sin A}{\sin BC} = \frac{\sin B}{\sin AC} = \frac{\sin C}{\sin AB} \ldots\ldots\ldots\ldots\ldots\ldots(\text{i.})$$

In the figure we have supposed the perpendicular from D on the opposite plane to fall *within* the angle BOC,—if it fall *without* it, the proposition admits of a proof in the same words.

29. *To express the cosine of an angle of a triangle in terms of the cosines and sines of the sides.*

From any point D in OA draw DE and DF in the planes AOB and AOC at right angles to AO. Therefore $\angle EDF$ is the inclination of those planes to each other, that is, the angle A. Join EF.



Then from the triangles FOE and FDE,

$$OF^2 + OE^2 - 2\,OF \,.\, OE \,.\, \cos FOE = FE^2$$

$$= FD^2 + DE^2 - 2\,FD \,.\, DE \,.\, \cos FDE\,;$$

$$\therefore\ 2\,OF \,.\, OE \,.\, \cos FOE$$

$$= (OF^2 - FD^2) + (OE^2 - DE^2) + 2\,FD \,.\, DE \,.\, \cos FDE$$

$$= 2\,OD^2 + 2 \,.\, FD \,.\, DE \,.\, \cos FDE\,;$$

$$\therefore\ \cos FOE = \frac{OD}{OF} \,.\, \frac{OD}{OE} + \frac{FD}{OF} \,.\, \frac{DE}{OE} \,.\, \cos FDE\,;$$

$$\therefore\ \cos a = \cos b \,.\, \cos c + \sin b \,.\, \sin c \,.\, \cos A \ldots\ldots\ldots(\text{ii.})$$

$$\text{Whence,}\quad \cos A = \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c}.$$

Cor. By writing $\pi - a$ for A, &c. (Art. 19), we have

$$\cos A + \cos B \,.\, \cos C = \sin B \,.\, \sin C \,.\, \cos a \text{ ....} \quad \text{(iii.)}$$

Whence,

$$\cos a = \frac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C}.$$

30. *To shew that*

$$\text{Sin c . cos A} = \text{cos a . sin b} - \text{sin a . cos b . cos C}.$$

By (ii),

$$\cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C.$$

So also,

$$\begin{aligned}\sin b \,.\, \sin c \,.\, \cos A &= \cos a - \cos b \,.\, \cos c \\ &= \cos a - \cos b \,.\, (\cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C) \\ &= \cos a \,.\, \left\{1 - (\cos b)^2\right\} - \cos b \,.\, \sin a \,.\, \sin b \,.\, \cos C;\end{aligned}$$

$$\therefore \quad \sin c \,.\, \cos A = \cos a \,.\, \sin b - \sin a \,.\, \cos b \,.\, \cos C \text{.........} \text{(iv.)}$$

Cor. 1.

$$\text{Since,} \quad \sin c \,.\, \cos A = \sin a \,.\, \frac{\sin C}{\sin A} \,.\, \cos A$$

$$= \sin a \,.\, \sin C \,.\, \cot A,$$

we have from (iv.),

$$\sin a \,.\, \sin C \,.\, \cot A = \cos a \,.\, \sin b - \sin a \,.\, \cos b \,.\, \cos C;$$

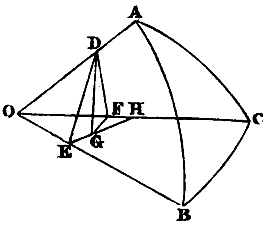
$$\therefore \quad \cot a \,.\, \sin b = \cos b \,.\, \cos C + \sin C \,.\, \cot A \text{...........} \text{(v.)}$$

Cor. 2. By writing $\pi - a$ for A, &c. (Art. 19.), we have from (iv.),

$$\sin C \,.\, \cos a = \cos A \,.\, \sin B + \sin A \,.\, \cos B \,.\, \cos c \,\ldots\ldots\ldots \text{(vi.)}$$

31. *To prove the formula* (iv.) *by an independent method.*

Taking the construction of Art. 28, produce EG to meet OC in H.



Then $\sin a = \dfrac{EH}{OH} = \dfrac{EG}{OH} + \dfrac{GH}{OH}$

$= \dfrac{EG}{OH} + \dfrac{GF}{OE}$, by △ OEH, GFH

$= \dfrac{EG}{ED} \cdot \dfrac{ED}{EO} \cdot \dfrac{EO}{OH} + \dfrac{GF}{FD} \cdot \dfrac{FD}{OD} \cdot \dfrac{OD}{OE}$ *

$= \cos B \,.\, \dfrac{\sin c}{\cos c} \,.\, \cos a + \cos C \,.\, \sin b \,.\, \dfrac{1}{\cos c}$.

$$\therefore \quad \sin b \,.\, \cos C = \cos c \,.\, \sin a - \sin c \,.\, \cos a \,.\, \cos B,$$

which is the same as (iv.).

---

* Note. The quantities introduced into the numerator and denominator of $\dfrac{EG}{OH}$, are those which connect EG and OH by means of the sides which are common to each two of the triangles DEG, DOE, OEH. Thus by introducing ED we connect the triangle DEG, in which EG is, with the triangle DEO; and by introducing EO we connect DEO with OEH, the triangle in which OH is. In like manner GF and OE are connected through the triangles GFD, FDO, DOE.

Cor. Hence (ii.) can be proved without taking for granted the Proposition in Plane Trigonometry,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

For $\cos a \cdot \sin c \cdot \cos B = \cos c \cdot \sin a - \sin b \cdot \cos C.$

So $\cos a \cdot \sin b \cdot \cos C = \cos b \cdot \sin a - \sin c \cdot \cos B,$

and substituting this value of $\sin c \cdot \cos B$ in the first equation,

$$\cos a \cdot (\cos b \cdot \sin a - \cos a \cdot \sin b \cdot \cos C) = \cos c \cdot \sin a - \sin b \cdot \cos C;$$

$$\therefore \quad \cos C \cdot \left\{1 - (\cos a)^2\right\} \cdot \sin b = \cos c \cdot \sin a - \cos a \cdot \cos b \cdot \sin a;$$

$$\therefore \quad \cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C.$$

32. The following theorem, was first given in this form by Gauss.

If, $p = \cos\frac{c}{2} \cdot \sin\frac{1}{2}(A + B)$, $\quad P = \cos\frac{C}{2} \cdot \cos\frac{1}{2}(a - b)$,

$q = \cos\frac{c}{2} \cdot \cos\frac{1}{2}(A + B)$, $\quad Q = \sin\frac{C}{2} \cdot \cos\frac{1}{2}(a + b)$,

$r = \sin\frac{c}{2} \cdot \sin\frac{1}{2}(A - B)$, $\quad R = \cos\frac{C}{2} \cdot \sin\frac{1}{2}(a - b)$,

$s = \sin\frac{c}{2} \cdot \cos\frac{1}{2}(A - B)$, $\quad S = \sin\frac{C}{2} \cdot \sin\frac{1}{2}(a + b)$,

Then the products $\mathrm{p} \cdot \mathrm{q}$, $\mathrm{p} \cdot \mathrm{r}$, $\mathrm{p} \cdot \mathrm{s}$, $\mathrm{q} \cdot \mathrm{r}$, $\mathrm{q} \cdot \mathrm{s}$, $\mathrm{r} \cdot \mathrm{s}$ *are respectively equal to the products* $\mathrm{P} \cdot \mathrm{Q}$, $\mathrm{P} \cdot \mathrm{R}$, $\mathrm{P} \cdot \mathrm{S}$, $\mathrm{Q} \cdot \mathrm{R}$, $\mathrm{Q} \cdot \mathrm{S}$, $\mathrm{R} \cdot \mathrm{S}$.

I. $$1+\frac{\sin B}{\sin A}=1+\frac{\sin b}{\sin a};$$

$$\therefore \quad \sin a\,.\,(\sin A+\sin B)=\sin A\,.\,(\sin a+\sin b);$$

$$\therefore \quad \sin c\,.\,(\sin A+\sin B)=\sin C\,.\,(\sin a+\sin b)\ldots\ldots(1).$$

Similarly, from $1-\frac{\sin B}{\sin A}=1-\frac{\sin b}{\sin a}$,

we should get

$$\sin c\,.\,(\sin A-\sin B)=\sin C\,.\,(\sin a-\sin b)\ldots\ldots(2).$$

II. By (iv.),

$$\sin c\,.\,\cos A=\cos a\,.\,\sin b-\sin a\,.\,\cos b\,.\,\cos C.$$

So also $\sin c\,.\,\cos B=\cos b\,.\,\sin a-\sin b\,.\,\cos a\,.\,\cos C$;

$$\therefore\ \sin c\,.\,(\cos B+\cos A)=(\sin a\,.\cos b+\cos a\,.\sin b)\,.\,(1-\cos C)$$
$$=\sin(a+b)\,.\,(1-\cos C)\ldots\ldots\ldots\ldots\ldots(3).$$

Similarly,

$$\sin c\,.\,(\cos B-\cos A)=\sin(a-b)\,.\,(1+\cos C)\ \ldots\ldots\ (4).$$

III. Writing $\pi-C$, &c. for c, &c. (Art. 19), and transposing the terms of (3) and (4), we have

$$(1+\cos c)\,.\,\sin(A+B)=\sin C\,.\,(\cos b+\cos a)\ \ldots\ldots\ (5).$$

$$(1-\cos c)\,.\,\sin(A-B)=\sin C\,.\,(\cos b-\cos a)\ \ldots\ldots\ (6).$$

And expressing these six forms in terms of the sines and cosines of

$$\frac{C}{2},\quad \frac{c}{2},\quad \tfrac{1}{2}(A\pm B),\quad \tfrac{1}{2}(a\pm b),$$

they become,

From I. $$\left\{\begin{array}{l} \sin\frac{c}{2}\,.\,\cos\frac{c}{2}\,.\,\sin\tfrac{1}{2}(A+B)\,.\,\cos\tfrac{1}{2}(A-B) = \sin\frac{C}{2}\,.\,\cos\frac{C}{2}\,.\,\sin\tfrac{1}{2}(a+b)\,.\,\cos\tfrac{1}{2}(a-b)\ldots\ldots(1). \\ \sin\frac{c}{2}\,.\,\cos\frac{c}{2}\,.\,\sin\tfrac{1}{2}(A-B)\,.\,\cos\tfrac{1}{2}(A+B) = \sin\frac{C}{2}\,.\,\cos\frac{C}{2}\,.\,\sin\tfrac{1}{2}(a-b)\,.\,\cos\tfrac{1}{2}(a+b)\ldots\ldots(2). \end{array}\right.$$

From II. $$\left\{\begin{array}{l} \sin\frac{c}{2}\,.\,\cos\frac{c}{2}\,.\,\cos\tfrac{1}{2}(A+B)\,.\,\cos\tfrac{1}{2}(A-B) = \sin\frac{C}{2}\,.\,\sin\frac{C}{2}\,.\,\sin\tfrac{1}{2}(a+b)\,.\,\cos\tfrac{1}{2}(a+b)\ldots\ldots(3). \\ \sin\frac{c}{2}\,.\,\cos\frac{c}{2}\,.\,\sin\tfrac{1}{2}(A+B)\,.\,\sin\tfrac{1}{2}(A-B) = \cos\frac{C}{2}\,.\,\cos\frac{C}{2}\,.\,\sin\tfrac{1}{2}(a-b)\,.\,\cos\tfrac{1}{2}(a-b)\ldots\ldots(4). \end{array}\right.$$

From III. $$\left\{\begin{array}{l} \cos\frac{c}{2}\,.\,\cos\frac{c}{2}\,.\,\sin\tfrac{1}{2}(A+B)\,.\,\cos\tfrac{1}{2}(A+B) = \sin\frac{C}{2}\,.\,\cos\frac{C}{2}\,.\,\cos\tfrac{1}{2}(a+b)\,.\,\cos\tfrac{1}{2}(a-b)\ldots\ldots(5). \\ \sin\frac{c}{2}\,.\,\sin\frac{c}{2}\,.\,\sin\tfrac{1}{2}(A-B)\,.\,\cos\tfrac{1}{2}(A-B) = \sin\frac{C}{2}\,.\,\cos\frac{C}{2}\,.\,\sin\tfrac{1}{2}(a+b)\,.\,\sin\tfrac{1}{2}(a-b)\ldots\ldots(6). \end{array}\right.$$

which six equations are,

$$\left\{\begin{array}{l} p\,.\,s = P\,.\,S \\ q\,.\,r = Q\,.\,R \end{array}\right\}\ldots\ldots\ldots\left\{\begin{array}{l} q\,.\,s = Q\,.\,S \\ p\,.\,r = P\,.\,R \end{array}\right\}\ldots\ldots\ldots\left\{\begin{array}{l} p\,.\,q = P\,.\,Q \\ r\,.\,s = R\,.\,S \end{array}\right\}.$$

33. *The quantities* p, q, r, s *are respectively equal to* P, Q, R, S.

$$\text{Since}\quad pq\,.\,pr = PQ\,.\,PR\,;$$

$$\therefore\quad p^2\,.\,qr = P^2\,.\,QR\,;$$

$$\therefore\quad p^2 = P^2.$$

$$\text{Similarly,}\qquad q^2 = Q^2,\quad r^2 = R^2,\quad s^2 = S^2\,;$$

$$\therefore\quad p = \pm P,\quad q = \pm Q,\quad r = \pm R,\quad s = \pm S.$$

Now in a spherical triangle, Art. 27.,

$\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$, are each less than π,

$\frac{1}{2}(A-B)$ and $\frac{1}{2}(a-b)$, are each less than $\frac{\pi}{2}$,

and are of the same sign,

$\frac{C}{2}$ and $\frac{c}{2}$ are each less than $\frac{\pi}{2}$;

$\therefore$ p, r, s, P, R, S are all positive quantities.

Also since $pq = PQ$, and $p = +P$, $\therefore$ $q = +Q$;

$$\therefore\ p = +P,\ \text{or}\ \cos\frac{c}{2}\,.\,\sin\tfrac{1}{2}(A+B) = \cos\frac{C}{2}\,.\,\cos\tfrac{1}{2}(a-b)\ldots\ldots\text{(vii.)}$$

$$q = +Q,\qquad \cos\frac{c}{2}\,.\,\cos\tfrac{1}{2}(A+B) = \sin\frac{C}{2}\,.\,\cos\tfrac{1}{2}(a+b)\ldots\text{(viii.)}$$

$$r = +R,\qquad \sin\frac{c}{2}\,.\,\sin\tfrac{1}{2}(A-B) = \cos\frac{C}{2}\,.\,\sin\tfrac{1}{2}(a-b)\ldots\ldots\text{(ix.)}$$

$$s = +S,\qquad \sin\frac{c}{2}\,.\,\cos\tfrac{1}{2}(A-B) = \sin\frac{C}{2}\,.\,\sin\tfrac{1}{2}(a+b)\ldots\ldots\text{(x.)}$$

Cor. From (viii.) it appears that $\cos\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(a+b)$ are of the same sign, and therefore $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are both greater or both less than $\frac{\pi}{2}$.

Def. When two angles are both greater or both less than a right angle, they are said to be *of the same*, or *of like*, *affection*.

Thus $\frac{1}{2}(A+B)$ and $\frac{1}{2}(a+b)$ are of like affection,—a property of spherical triangles which may be added to those enumerated in Art. 27.

34. Since

$$\frac{p}{q}=\frac{P}{Q}; \quad \therefore \tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}.\cot\frac{C}{2}\ldots\ldots(\text{xi.})$$

$$\frac{r}{s}=\frac{R}{S}; \quad \therefore \tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.\cot\frac{C}{2}\ldots\ldots(\text{xii.})$$

$$\frac{S}{Q}=\frac{s}{q}; \quad \therefore \tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}.\tan\frac{c}{2}\ldots\ldots(\text{xiii.})$$

$$\frac{R}{P}=\frac{r}{p}; \quad \therefore \tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}.\tan\frac{c}{2}\ldots\ldots(\text{xiv.})$$

These four equations are called Napier's analogies, and are of great use in the solution of triangles*.

---

* The quantities p, q, r, s, P, Q, R, S, will be most easily remembered from Napier's first and second analogies written thus,

$$\frac{p}{q}=\frac{\sin\frac{1}{2}(A+B).\cos\frac{c}{2}}{\cos\frac{1}{2}(A+B).\cos\frac{c}{2}}=\frac{\cos\frac{1}{2}(a-b).\cos\frac{C}{2}}{\cos\frac{1}{2}(a+b).\sin\frac{C}{2}}=\frac{P}{Q}.$$

$$\frac{r}{s}=\frac{\sin\frac{1}{2}(A-B).\sin\frac{c}{2}}{\cos\frac{1}{2}(A-B).\sin\frac{c}{2}}=\frac{\sin\frac{1}{2}(a-b).\cos\frac{C}{2}}{\sin\frac{1}{2}(a+b).\sin\frac{C}{2}}=\frac{R}{S}.$$

35. The following expressions can, without much difficulty, be proved from the formulæ of Art. 33; we will however establish them from the fundamental formulæ (i), (ii), (iii).

To find the values of $\cos\frac{A}{2}$, $\sin\frac{A}{2}$, $\tan\frac{A}{2}$, *and* $\sin A$, *in terms of* a, b, c.

By (ii),

$$\sin b \,.\, \sin c \,.\, \cos A = \cos a - \cos b \,.\, \cos c;$$

$$\therefore\ \sin b \,.\, \sin c \,.\, (1 + \cos A) = \cos a - (\cos b \,.\, \cos c - \sin b \,.\, \sin c);$$

$$\therefore\ \sin b \,.\, \sin c \,.\, 2\left(\cos\frac{A}{2}\right)^2 = \cos a - \cos(b + c)$$

$$= 2\sin\tfrac{1}{2}(a + b + c) \,.\, \sin\tfrac{1}{2}(b + c - a).$$

Pl. Trig. Art. 51, (4).

And if $S = \frac{1}{2}(a+b+c)$, $S - a = \frac{1}{2}(a + b + c) - a = \frac{1}{2}(b + c - a)$,

$$\therefore\ \left(\cos\frac{A}{2}\right)^2 = \frac{\sin S \,.\, \sin(S - a)}{\sin b \,.\, \sin c};$$

$$\therefore\ \cos\frac{A}{2} = \sqrt{\frac{\sin S \,.\, \sin(S - a)}{\sin b \,.\, \sin c}} \ \ldots\ldots\ (\text{xv.})$$

$$\text{Again, } 2\left(\sin\frac{A}{2}\right)^2 = 1 - \cos A = 1 - \frac{\cos a - \cos b \,.\, \cos c}{\sin b \,.\, \sin c};$$

$$\text{Whence, } \sin\frac{A}{2} = \sqrt{\frac{\sin(S - b) \,.\, \sin(S - c)}{\sin b \,.\, \sin c}} \ \ldots\ldots\ (\text{xvi.})$$

$$\text{Also, } \tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{\sin(S - b) \,.\, \sin(S - c)}{\sin S \,.\, \sin(S - a)}} \ \ldots\ldots\ (\text{xvii.})$$

$$\text{And } \sin A = 2\sin\frac{A}{2} \,.\, \cos\frac{A}{2}$$

$$= \frac{2}{\sin b \,.\, \sin c}\sqrt{\{\sin S \,.\, \sin(S-a) \,.\, \sin(S-b) \,.\, \sin(S-c)\}} \ \ldots(\text{xviii.})$$

The positive signs of the square roots are taken in these cases, because $\frac{A}{2}$ is necessarily less than $\frac{\pi}{2}$.

36. *To find the values of* $\cos\frac{a}{2}$, $\sin\frac{a}{2}$, $\tan\frac{a}{2}$, *and* $\sin a$, *in terms of* A, B, C.

By (iii.) we have, $\cos a = \dfrac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C}$;

$$\therefore\ 2\left(\cos\frac{a}{2}\right)^2 = 1 + \cos a = \frac{\cos A + \cos B \,.\, \cos C + \sin B \,.\, \sin C}{\sin B \,.\, \sin C}$$

$$= \frac{\cos A + \cos(B - C)}{\sin B \,.\, \sin C}$$

$$= \frac{2\cos\frac{1}{2}(A+C-B) \,.\, \cos\frac{1}{2}(A+B-C)}{\sin B \,.\, \sin C}.$$

And if $S' = \frac{1}{2}(A + B + C)$; then, $S' - B = \frac{1}{2}(A + C - B)$;

$$\therefore\ \cos\frac{a}{2} = \sqrt{\frac{\cos(S' - B) \,.\, \cos(S' - C)}{\sin B \,.\, \sin C}} \ \ldots\ldots\ \text{(xx.)}$$

$$\text{Again,}\quad 2\left(\sin\frac{a}{2}\right)^2 = 1 - \cos a = 1 - \frac{\cos A + \cos B \,.\, \cos C}{\sin B \,.\, \sin C}$$

$$= -\frac{\cos A + \cos(B + C)}{\sin B \,.\, \sin C};$$

$$\text{Whence,}\quad \sin\frac{a}{2} = \sqrt{-\frac{\cos S' \,.\, \cos(S' - A)}{\sin B \,.\, \sin C}} \ \ldots\ldots\ldots\ldots\text{(xxi.)}$$

$$\text{Also,}\quad \tan\frac{a}{2} = \frac{\sin\frac{a}{2}}{\cos\frac{a}{2}} = \sqrt{-\frac{\cos S' \,.\, \cos(S' - A)}{\cos(S' - B) \,.\, \cos(S' - C)}} \ \ldots\ \text{(xxii.)}$$

$$\text{And}\quad \sin a = 2\sin\frac{a}{2} \,.\, \cos\frac{a}{2}$$

$$= \frac{2}{\sin B.\sin C}\sqrt{\left\{-\cos S'.\cos(S'-A).\cos(S'-B).\cos(S'-C)\right\}}\,\text{(xxiii.)}$$

Note. Since the angles of a spherical triangle are together greater than two right angles and less than six, S' is greater than one right angle and less than three, and its cosine is therefore a negative quantity. Also since, Art. 27, $A+B-C<\pi$, therefore $\cos\frac{1}{2}(A+B-C)$, or $\cos(S'-C)$, is a positive quantity; and in like manner $\cos(S'-B)$, and $\cos(S'-A)$, are positive: Wherefore the last three formulæ are not *imaginary* quantities, as at first sight they appear to be, but *real* quantities, as they ought to be.

37. Before leaving this part of the subject, we will prove Napier's Analogies from the formulæ (i.), (ii.), (iii).

By (iii.),

$$\cos A + \cos B \,.\, \cos C = \cos a \,.\, \sin B \,.\, \sin C.$$

So, $$\cos B + \cos A \,.\, \cos C = \cos b \,.\, \sin A \,.\, \sin C.$$

By addition, $$(\cos A + \cos B)\,.\,(1 + \cos C)$$

$$= (\cos a \,.\, \sin B + \cos b \,.\, \sin A)\,.\,\sin C;$$

$$\therefore\quad (\cos A + \cos B)\,.\,2\left(\cos\frac{C}{2}\right)^2$$

$$= \left(\cos a \,.\, \sin A \,.\, \frac{\sin b}{\sin a} + \cos b \,.\, \sin A\right).\,2\sin\frac{C}{2}\,.\,\cos\frac{C}{2};$$

$$\therefore\ \cos A + \cos B = \frac{\sin A}{\sin a}\,.\,(\cos a \,.\, \sin b + \cos b \,.\, \sin a)\,.\,\tan\frac{C}{2}$$

$$= \frac{\sin A}{\sin a}\,.\,\sin(a+b)\,.\,\tan\frac{C}{2};$$

Again; $\sin A + \sin B = \sin A \, . \left(1 + \frac{\sin B}{\sin A}\right)$

$$= \sin A \, . \left(1 + \frac{\sin b}{\sin a}\right)$$

$$= \frac{\sin A}{\sin a} \, . \, (\sin a + \sin b);$$

$$\therefore \quad \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin (a+b)} \, . \cot \frac{C}{2},$$

$$\frac{2 \, . \sin \tfrac{1}{2}(A+B) \, . \cos \tfrac{1}{2}(A-B)}{2 \, . \cos \tfrac{1}{2}(A+B) \, . \cos \tfrac{1}{2}(A-B)} = \frac{2 \, . \sin \tfrac{1}{2}(a+b) \, . \cos \tfrac{1}{2}(a-b)}{2 \, . \sin \tfrac{1}{2}(a+b) \, . \cos \tfrac{1}{2}(a+b)} \, . \cot \frac{C}{2};$$

$$\therefore \quad \tan \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2}(a+b)} \, . \cot \frac{C}{2} \, \ldots\ldots\ldots \, (\alpha).$$

In like manner,

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin a - \sin b}{\sin (a+b)} \, . \cot \frac{C}{2};$$

Whence $\tan \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2}(a+b)} \, . \cot \frac{C}{2} \, \ldots\ldots\ldots \, (\beta).$

By writing $\pi - A$ for a, &c., Art. 19, the two formulæ (α) and (β) become

$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \tfrac{1}{2}(A-B)}{\cos \tfrac{1}{2}(A+B)} \, . \tan \frac{c}{2} \ldots\ldots\ldots (\gamma).$$

$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}(A+B)} \, . \tan \frac{c}{2} \ldots\ldots\ldots (\delta).$$

These last two analogies might have been easily proved independently from the formula,

$$\cos a = \cos b \, . \cos c + \sin b \, . \sin c \, . \cos A.$$

CHAPTER III.

ON THE SOLUTION OF RIGHT-ANGLED TRIANGLES.

38. Right-angled triangles may in all cases,—with an exception which will be pointed out,—be solved by means of the following formulæ, when, besides the right angle, two other quantities out of the three sides and the two remaining angles are given.

If A, B, C be the angles of a spherical triangle, and a, b, c the sides respectively opposite to them, we have seen that

(ii.) $\cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C$ ....(1).

(iii.) $\cos c \,.\, \sin A \,.\, \sin B = \cos C + \cos A \,.\, \cos B$ ....(2).

$\cos a \,.\, \sin B \,.\, \sin C = \cos A + \cos B \,.\, \cos C$.........(3).

(i.) $\sin a \,.\, \sin C = \sin c \,.\, \sin A$ ....(4).

(v.) $\cot a \,.\, \sin b = \cos b \,.\, \cos C + \sin C \,.\, \cot A$ ....(5).

$\cot c \,.\, \sin a = \cos a \,.\, \cos B + \sin B \,.\, \cot C$ ....(6).

Let $C = 90^0$, and $\therefore \cos C = 0$, $\sin C = 1$, and we obtain from these equations,

(1) $\cos c = \cos a \,.\, \cos b.$

(2) $\cos c = \cot A \,.\, \cot B.$

(3) $\cos A = \cos a \,.\, \sin B.$

$\cos B = \cos b \,.\, \sin A.$

(4) $\sin a = \sin c \,.\, \sin A.$

$\sin b = \sin c \,.\, \sin B.$

(5) $\sin b = \tan a \,.\, \cot A.$

$\sin a = \tan b \,.\, \cot B.$

(6) $\cos B = \cot c \,.\, \tan a.$

$\cos A = \cot c \,.\, \tan b.$

39. NAPIER'S RULES.

There is a kind of artificial memory by which all the formulæ in the last Article are comprised under two rules, which take their name from Napier, who first gave them.

The right angle being left out of consideration, the *two sides including the right angle*, and the *complements of the hypothenuse* and *of the other angles*, are called the *circular parts* of the triangle. Any one of these being fixed upon as the middle part (M), the two circular parts *immediately* joining it are called the adjacent parts (A_1, A_2), and the other two parts are called the opposite parts, (O_1, O_2).

Thus in the triangle ABC whose right angle is C;

If M be	The adjacent Parts are	The opposite Parts are
a, (one of the sides including the right angle).	$\frac{\pi}{2}-B$, b.	$\frac{\pi}{2}-A$, $\frac{\pi}{2}-c$.
$\frac{\pi}{2}-A$, (the complement of an angle).	b, $\frac{\pi}{2}-c$.	$\frac{\pi}{2}-B$, a.
$\frac{\pi}{2}-c$, (the complement of the hypothenuse).	$\frac{\pi}{2}-A$, $\frac{\pi}{2}-B$.	a, b.

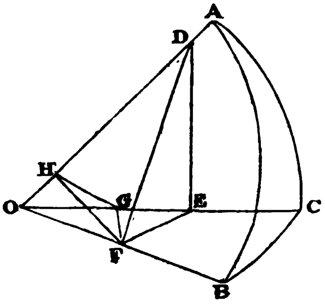
Now NAPIER'S RULES are,

$\sin M =$ *product of the tangents of the adjacent parts* $= \tan A_1 . \tan A_2$,

$\sin M =$ *product of the cosines of the opposite parts* $= \cos O_1 . \cos O_2$,

with which the formulæ of the last Article will be found, on trial, to agree.

40. In Art. 38, Napier's Rules have been deduced as corollaries from certain formulæ. They may however be proved independently in the following manner.



Let *ABC* be the triangle, $\angle C = 90^0$.

From *D*, any point in *AO*, draw *DE* perpendicular to *OC*, and let *DFE* be that plane passing through *DE* which *OB* cuts at right angles. Then $\angle DFE = \angle B$.

From *F*, draw *FG* perpendicular to *OC*, and let the plane *FGH*, passing through *FG*, cut *AO* perpendicularly. Then $\angle FHG = \angle A$.

Also *FEO*, *GFO* are similar triangles. So also are *DEO*, *GHO*.

$$\text{I.} \qquad \text{Sin}\, a = \frac{FE}{OE} = \frac{FE}{ED} \cdot \frac{ED}{OE} = \cot B \, . \, \tan b.$$

$$\text{Also,} \qquad \sin a = \sin A \, . \, \frac{\sin c}{\sin C} = \sin A \, . \, \sin c. \qquad \text{Art. 28.}$$

$$\text{II.} \qquad \text{Cos}\, B = \frac{FE}{FD} = \frac{FE}{OF} \cdot \frac{OF}{FD} = \tan a \, . \, \cot c.$$

$$\text{Also,} \qquad \cos B = \frac{FE}{FD} = \frac{OF \, . \, \dfrac{FG}{OG}}{OF \, . \, \dfrac{FH}{OH}} = \frac{FG}{FH} \cdot \frac{OH}{OG}$$

$$= \sin A \, . \, \cos b.$$

$$\text{III.} \qquad \text{Cos}\, c = \frac{OF}{OD} = \frac{OF}{OE} \cdot \frac{OE}{OD} = \cos a \, . \, \cos b.$$

$$\text{Also,} \qquad \cos c = \frac{OF}{OD} = \frac{FE \, . \, \dfrac{OG}{GF}}{DE \, . \, \dfrac{OG}{GH}} = \frac{FE}{DE} \cdot \frac{GH}{GF}$$

$$= \cot B \, . \, \cot A.$$

From I. and II. we have, writing b, B for a, A, and conversely,

$$\sin b = \cot A \, . \tan a, \qquad \cos A = \tan b \, . \cot c,$$
$$= \sin B \, . \sin c, \qquad = \sin B \, . \cos a.$$

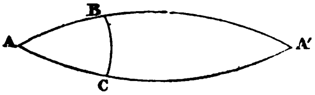
41. *To shew that in a triangle* ABC, *in which* C *is a right angle*, A *and* a *are of the same affection, as are also* B *and* b.

$$\text{Sin}\, b = \cot A \, . \tan a = \frac{\tan a}{\tan A}.$$

Now since b is less than π, $\sin b$ is positive;—therefore $\tan a$ and $\tan A$ must be of the same sign. And because π is the limit both of a and of A, these angles must be both greater or both less than a right angle; that is, A and a are of the same affection.

Similarly, from $\sin a = \dfrac{\tan b}{\tan B}$, it appears that B and b are of the same affection.

42. *If in a right-angled triangle, an angle and the side opposite to it be the only given quantities, the triangle cannot be determined.*



For if the circles AB and AC intersect again in A', and C be a right angle, it is evident that ACB and $A'CB$ have the angles A, A' equal, and CB the side opposite to these angles is the same in both triangles. It is therefore ambiguous whether ABC or $A'BC$ be the triangle sought.

This ambiguity will also appear, if it be attempted to determine the triangle by Napier's rules. For we cannot tell from the equation

$$\sin AC = \tan CB \, . \tan\left(\frac{\pi}{2} - A\right),$$

whether we are to take the angle AC, or its supplement $A'C$.

43. The solutions of the other cases of a right-angled triangle from two given parts are not ambiguous, if attention be paid to the two following principles,

(1). The greater side is opposite to the greater angle.

(2). An angle and the side opposite to it are of the same affection.

For example: Let c and A be given, to find a, B, b.

$$\text{Sin}\, a = \cos\left(\frac{\pi}{2} - A\right) . \cos\left(\frac{\pi}{2} - c\right) = \sin A . \sin c.$$

And since a and A are of the same affection, the greater or lesser angle which satisfies this equation is to be taken for a, according as A is greater or less than $\frac{\pi}{2}$.

$$\text{Again, } \sin\left(\frac{\pi}{2} - c\right), \text{ or } \cos c, = \cot A . \cot B;$$

$$\therefore \tan B = \cot A . \sec c.$$

And B is $\lesseqgtr 90^0$ as the second member of the equation is positive or negative; that is, as A and c are of like or unlike affection.

$$\text{Again, } \sin\left(\frac{\pi}{2} - c\right), \text{ or } \cos c, = \cos a . \cos b;$$

$$\therefore \cos b = \cos c . \sec a.$$

And b is $\lesseqgtr 90^0$, according as the second member of the equation is positive or negative, that is, as a and c are of like or unlike affection.

44. In selecting a formula attention must be paid to the principles laid down in Appendix II. to Pl. Trig. The following formulæ may be used with advantage, when the side or angle required is small, or nearly equal to one or to two right angles.

By Napier's rules, $\cos c = \cot A . \cot B$, whence c cannot be accurately determined, if it be either a very small angle or nearly equal to two right angles.

Now $2\left(\sin\frac{c}{2}\right)^2 = 1 - \cos c = 1 - \cot A \,.\, \cot B = 1 - \frac{\cos A \,.\, \cos B}{\sin A \,.\, \sin B}$;

$$\therefore \; \sin\frac{c}{2} = \sqrt{\frac{-\cos(A+B)}{2\sin A \,.\, \sin B}}.$$

$$\text{So } \cos\frac{c}{2} = \sqrt{\tfrac{1}{2}(1+\cos c)} = \sqrt{\frac{\cos(A-B)}{2\sin A \,.\, \sin B}}.$$

In like manner from $\cos a = \dfrac{\sin\left(\frac{\pi}{2} - A\right)}{\sin B}$, we have

$$\cos\frac{a}{2} = \sqrt{\frac{\sin\left\{\frac{1}{2}(B-A)+\frac{\pi}{4}\right\} . \cos\left\{\frac{1}{2}(B+A)-\frac{\pi}{4}\right\}}{\sin B}};$$

$$\sin\frac{a}{2} = \sqrt{\frac{\sin\left\{\frac{1}{2}(B+A)-\frac{\pi}{4}\right\} . \cos\left\{\frac{1}{2}(B-A)+\frac{\pi}{4}\right\}}{\sin B}}.$$

So from $\cos A = \cot c \,.\, \tan b$,

$$\sin\frac{A}{2} = \sqrt{\tfrac{1}{2}(1-\cot c \,.\, \tan b)} = \sqrt{\frac{\sin(c-b)}{2\sin c \,.\, \cos b}},$$

$$\cos\frac{A}{2} = \sqrt{\tfrac{1}{2}(1+\cot c \,.\, \tan b)} = \sqrt{\frac{\sin(c+b)}{2\sin c \,.\, \cos b}}.$$

When c and A are given, if a be nearly a right angle, it cannot be accurately determined from its sine. In this case $\cot B = \cos c \,.\, \tan A$ determines B, and a may be found from the formula for $\sin\frac{a}{2}$, or from that for $\cos\frac{a}{2}$. Art. 36.

45. **Def.** A triangle is called *quadrantal*, if any one of its sides be a quadrant.

A quadrantal triangle ABC, *of which the side* c *is a quadrant, may be solved by Napier's rules, if the quadrantal side be neglected, and* $\frac{\pi}{2} - a$, $\frac{\pi}{2} - b$, A, B, $C - \frac{\pi}{2}$, *be taken for the circular parts.*

Let $A'B'C'$ be the polar triangle. It may be solved by Napier's rules, because $C'\left(= \pi - c = \frac{\pi}{2}\right)$ is a right angle;

$$\therefore \left\{ \begin{array}{lll} \cos c' = \cot A' . \cot B', & \text{or} & = \cos a' . \cos b', \\ \sin a' = \tan b' . \cot B', & \text{or} & = \sin A' . \sin c', \\ \cos A' = \tan b' . \cot c', & \text{or} & = \cos a' . \sin B'. \end{array} \right.$$

Substituting for A, a, &c. $\pi - A$, $\pi - a$, &c. these become,

$$\left\{ \begin{array}{lll} -\cos C = \cot a . \cot b, & \text{or} & = \cos A . \cos B, \\ \sin A = \tan B . \tan b, & \text{or} & = \sin a . \sin C, \\ -\cos a = \tan B . \cot C, & \text{or} & = -\cos A . \sin b, \end{array} \right.$$

which may be represented thus,

$$\left\{ \begin{array}{lll} \sin\left(C - \frac{\pi}{2}\right) = \tan\left(\frac{\pi}{2} - a\right) . \tan\left(\frac{\pi}{2} - b\right), & \text{or} & = \cos A . \cos B, \\ \sin A = \tan B . \tan\left(\frac{\pi}{2} - b\right), & \text{or} & = \cos\left(\frac{\pi}{2} - a\right) . \cos\left(C - \frac{\pi}{2}\right), \\ \sin\left(\frac{\pi}{2} - a\right) = \tan B . \tan\left(C - \frac{\pi}{2}\right), & \text{or} & = \cos A . \cos\left(\frac{\pi}{2} - b\right), \end{array} \right.$$

which was to be proved.

A Collection of Examples is added at the end of this Treatise.

CHAPTER IV.

ON THE SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

46. *Let the three sides be given*, (a, b, c).

The angles may be determined from one of the formulæ (xvi.), (xvii.), (xviii.), (xix.).

47. *Let the three angles be given*, (A, B, C).

The sides may be determined from one of the formulæ (xx.), (xxi.), (xxii.), (xxiii.).

48. *Let two sides and the included angle be given*, (a, C, b).

By Napier's first and second analogies, (xii.) and (xiii.),

$$\left\{\begin{aligned} \tan\tfrac{1}{2}(A+B) &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cdot\cot\frac{C}{2}, \\ \tan\tfrac{1}{2}(A-B) &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cdot\cot\frac{C}{2}, \end{aligned}\right.$$

$\frac{1}{2}(A+B)$ and $\frac{1}{2}(A-B)$ are determined;

$$\therefore \left\{\begin{aligned} A &= \tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B), \\ B &= \tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B), \end{aligned}\right\} \text{ are known.}$$

And A and B being known, c is found from

$$\sin c = \sin a\cdot\frac{\sin C}{\sin A}.$$

49. *To determine* c *independently of* A *and* B, *by forms adapted to logarithmic computation.*

$$\text{Cos}\, c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C$$

$$= \cos b \,.\, (\cos a + \sin a \,.\, \tan b \,.\, \cos C).$$

Let θ be an angle such that

$$\tan\theta = \tan b \,.\, \cos C \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then $$\cos c = \cos b \,.\left(\cos a + \sin a \,.\, \frac{\sin\theta}{\cos\theta}\right)$$

$$= \frac{\cos b}{\cos\theta} \,.\, \cos(a-\theta) \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1), $L\tan\theta = L\tan b + L\cos C - 10$; which gives θ.

(2), $L\cos c = L\cos b + L\cos(a-\theta) - L\cos\theta$; which gives c.

50. *Let two angles and the included side be given,* (A, c, B).

From Napier's third and fourth analogies, (xiv.) and (xv.),

$$\begin{cases} \tan\frac{1}{2}(a+b) = \dfrac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} \,.\, \tan\dfrac{c}{2}, \\[2ex] \tan\frac{1}{2}(a-b) = \dfrac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)} \,.\, \tan\dfrac{c}{2}, \end{cases}$$

$\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$ are determined;

$$\therefore \left.\begin{cases} a = \frac{1}{2}(a+b) + \frac{1}{2}(a-b), \\ b = \frac{1}{2}(a+b) - \frac{1}{2}(a-b), \end{cases}\right\} \text{are known.}$$

And a and b being known, C is found from

$$\sin C = \sin A \,.\, \frac{\sin c}{\sin a}.$$

51. *To find* C *independently of* a *and* b, *by forms adapted to logarithmic computation.*

$$\operatorname{Cos} C = \cos c \,.\, \sin A \,.\, \sin B - \cos A \,.\, \cos B\,;$$

$$\therefore\ 2\left(\cos\frac{C}{2}\right)^2 - 1 = \left\{1 - 2\left(\sin\frac{c}{2}\right)^2\right\} . \sin A \,.\, \sin B - \cos A \,.\, \cos B$$

$$= -\cos(A+B) - 2\left(\sin\frac{c}{2}\right)^2 . \sin A \,.\, \sin B\,;$$

$$\therefore\ 2\left(\cos\frac{C}{2}\right)^2 = 1 - \cos(A+B) - 2\left(\sin\frac{c}{2}\right)^2 . \sin A \,.\, \sin B\,;$$

$$\therefore\ \left(\cos\frac{C}{2}\right)^2 = \left\{\sin\tfrac{1}{2}(A+B)\right\}^2 - \left(\sin\frac{c}{2}\right)^2 . \sin A \,.\, \sin B.$$

Now $\left(\sin\frac{c}{2}\right)^2 . \sin A \,.\, \sin B$ is necessarily positive, and less than unity. Let therefore θ be an angle such that

$$(\sin\theta)^2 = \left(\sin\frac{c}{2}\right)^2 . \sin A \,.\, \sin B \text{ ....(1)};$$

$$\therefore\ \left(\cos\frac{C}{2}\right)^2 = \left\{\sin\tfrac{1}{2}(A+B)\right\}^2 - (\sin\theta)^2$$

$$= \sin\left\{\tfrac{1}{2}(A+B)+\theta\right\} . \sin\left\{\tfrac{1}{2}(A+B)-\theta\right\}. \text{ Pl. Trig. Art. 54...(2).}$$

From (1), $L\sin\theta = \tfrac{1}{2}(L\sin A + L\sin B) + L\sin\frac{c}{2} - 10$; which gives θ.

From (2), $L\cos\frac{C}{2} = \tfrac{1}{2}\left[L\sin\left\{\tfrac{1}{2}(A+B)+\theta\right\} + L\sin\left\{\tfrac{1}{2}(A+B)-\theta\right\}\right]$; which gives C.

52. *Let two angles and a side opposite to one of them be given*, (A, B, a).

$$\text{Sin}\, b = \sin a \,.\, \frac{\sin B}{\sin A}.$$

$$\text{From (xii.),}\quad \tan\frac{C}{2} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} \,.\, \cot\tfrac{1}{2}(A+B).$$

$$\text{From (xiv.),}\quad \tan\frac{c}{2} = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} \,.\, \tan\tfrac{1}{2}(a+b).$$

And b having been determined from the first of these equations, C and c may be found from the last two.

53. *To determine* C *and* c *independently of* b, *by forms adopted to logarithmic computation.*

$$\text{Cos}\, A = \cos a \,.\, \sin B \,.\, \sin C - \cos B \,.\, \cos C$$

$$= \cos B \,.\, (\cos a \,.\, \tan B \,.\, \sin C - \cos C).$$

Let θ be an angle such that

$$\cot\theta = \cos a \,.\, \tan B \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)};$$

$$\therefore\quad \cos A = \cos B \,.\, \left(\frac{\cos\theta}{\sin\theta} \,.\, \sin C - \cos C\right)$$

$$= \frac{\cos B}{\sin\theta} \,.\, \sin(C-\theta) \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

From (1), $L\cot\theta = L\cos a + L\tan B - 10$; which gives θ.

(2), $L\sin(C-\theta) = L\cos A + L\sin\theta - L\cos B$; which gives $C-\theta$, and thence C.

Again, we have from (vii.),

$$\cot a \,.\, \sin c = \cos c \,.\, \cos B + \sin B \,.\, \cot A;$$

$$\therefore\quad \sin B \,.\, \cot A = \cot a \,.\, \sin c - \cos c \,.\, \cos B$$

$$= \cot a \,.\, (\sin c - \cos B \,.\, \tan a \,.\, \cos c),$$

whence, if θ be such that

$$\tan\phi = \cos B \,.\, \tan a, \dots\dots\dots\dots\dots\dots\dots\dots (1),$$

we have

$$\sin B \,.\, \cot A = \frac{\cot a}{\cos\phi} \,.\, \sin(c-\phi) \dots\dots\dots\dots\dots\dots (2).$$

From (1), $L \tan\phi = L \cos B + L \tan a - 10$; which gives ϕ.

(2), $L \sin(c-\phi) = L \sin B + L \cot A + L \cos\phi - L \cot a - 10$; which gives $c - \phi$, and thence c.

54. *Let two sides and an angle opposite to one of them be given*, (a, b, A).

$$\text{Sin } B = \sin A \,.\, \frac{\sin b}{\sin a},$$

$$\tan\frac{C}{2} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} \,.\, \cot\tfrac{1}{2}(A+B),$$

$$\tan\frac{c}{2} = \frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} \,.\, \tan\tfrac{1}{2}(a+b).$$

And B having been determined from the first equation, C and c are found from the last two equations.

55. *To determine* C *and* c *independently of* B, *by forms adapted to logarithmic computation.*

From (v.)

$$\cot a \,.\, \sin b = \cos b \,.\, \cos C + \sin C \,.\, \cot A$$

$$= \cos b \,.\, \left(\cos C + \frac{\cot A}{\cos b} \,.\, \sin C\right);$$

Whence, if θ be such that

$$\cot\theta = \frac{\cot A}{\cos b} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

we have

$$\sin(C+\theta) = \cot a \, . \tan b \, . \sin\theta \, ; \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

from which two equations we can determine θ and C.

Again, $\cos a = \cos b \, . \cos c + \sin b \, . \sin c \, . \cos A$

$$= \cos b \, . (\cos A \, . \tan b \, . \sin c + \cos c) \, ;$$

Whence, if ϕ be such that

$$\cot\phi = \cos A \, . \tan b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

we have

$$\sin(c+\phi) = \frac{\cos a \, . \sin\phi}{\cos b} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

And from these two equations we can determine ϕ and c.

56. In Article 52, the triangle has been apparently solved when two angles and a side opposite to one of them are given; and in Art. 54, the triangle has been apparently solved from two sides and the angle opposite to one of them.

It is, however, doubtful in some cases, whether in determining one of the quantities A, B, a, b, from the equation $\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$, when the other three are given, we ought to take an angle which is less than $\frac{\pi}{2}$, or one which is greater than $\frac{\pi}{2}$. We shall now proceed to distinguish the ambiguous cases from those which are not so.

(1) *Given* A, B, b, *to determine in what cases* a *may be found from the equation,*

$$\frac{\sin \mathrm{a}}{\sin \mathrm{b}} = \frac{\sin \mathrm{A}}{\sin \mathrm{B}}.$$

I. Let $A + B$ be greater than two right angles, or π.

Then since, Art. 33, Cor., $\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are of like affection,

$$\therefore\ a + b \text{ is } > \pi,$$

$$\text{or}\quad a > \pi - b.$$

If $\therefore$ b be $<\frac{\pi}{2}$, a is $>\frac{\pi}{2}$. In this case, therefore, a may be found from $\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$, and the triangle may be determined.

If b be $>\frac{\pi}{2}$, we have nothing to enable us to determine from the relation $a > \pi - b$ whether a is $>$ or $<\pi$.

II. If $A + B$ be $<\pi$, then $a + b$ is $<\pi$, and a is $<\pi - b$.

And a is therefore $<\frac{\pi}{2}$ if b be $>\frac{\pi}{2}$, but cannot be determined if b be $<\frac{\pi}{2}$.

III. If $A + B = \pi$; then, Art. 33, (ix.), $a + b = \pi$, and $\therefore$ $a = \pi - b$.

It appears then, that A, B, b being given,

$$\text{When}\begin{cases} A + B > \pi, \text{ and } b < \frac{\pi}{2},\ a \text{ is greater than } \frac{\pi}{2}, \\ A + B < \pi, \text{ and } b > \frac{\pi}{2},\ a \text{ is less than } \frac{\pi}{2}, \\ A + B = \pi;\ a = \pi - b. \end{cases}$$

And in no other case can a be found, and the triangle determined.

(2) *Given two sides, and an angle opposite to one of them, to determine when the remaining parts of the triangle may be determined.* (a, b, B.).

If $a + b \overset{>}{\underset{<}{=}} \pi$, then as before $A + B \overset{>}{\underset{<}{=}} \pi$, and $\therefore A \overset{>}{\underset{<}{=}} \pi - B$.

Whence we collect, as in the last proposition,

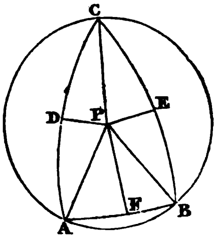
$$\text{When} \begin{cases} a + b > \pi, \text{ and } B < \dfrac{\pi}{2};\ A \text{ is greater than } \dfrac{\pi}{2}, \\ a + b < \pi, \text{ and } B > \dfrac{\pi}{2};\ A \text{ is less than } \dfrac{\pi}{2}, \\ a + b = \pi;\quad A = \pi - B. \end{cases}$$

And in no other case can A be determined.

57. By letting fall a perpendicular from an angle upon the opposite side, an oblique-angled triangle may be divided into two right-angled triangles, which may in most cases be solved by Napier's rules. The same ambiguities will however arise in this method of solution as have been shewn to exist when this construction is not used for the determination of the triangle. See Art. 42.

58. Prob. *To find the radius of a small circle described about a given triangle in terms of the* angles *of the triangle.*

Let ABC be the triangle; bisect CA and CB in D and E, and draw at right angles to AC and CB the arcs DP and EP intersecting in P. Join PA, PB, PC, by arcs of great circles.



Then, from the right-angled triangles PCD and PAD,

$$\cos PC = \cos PD \,.\, \cos DC = \cos PD \,.\, \cos DA = \cos PA;$$

$$\therefore PA = PC. \quad \text{Similarly, } PB = PC.$$

Therefore P is the pole of the circumscribing circle.

Let $PA = \rho$. Now $\cos PBE = \cot PB \,.\, \tan BE = \cot\rho \,.\, \tan\frac{a}{2}$;

$$\therefore\ \cot\rho = \cos PBE \,.\, \cot\frac{a}{2}.$$

And since PAC, PBC, PAB are isosceles triangles,

$$\therefore\ 2\angle PBE + 2\angle PAC + 2\angle PAB = A + B + C = 2S';$$

$$\therefore\ \angle PBE = S' - (\angle PAC + \angle PAB) = S' - A.$$

Also by (xxii.), $\cot\frac{a}{2} = \sqrt{\frac{\cos(S'-B)\,.\,\cos(S'-C)}{-\cos S'\,.\,\cos(S'-A)}}$;

$$\therefore\ \cot\rho = \sqrt{\frac{\cos(S'-A)\,.\,\cos(S'-B)\,.\,\cos(S'-C)}{-\cos S'}}.$$

59. *To determine the radius of the circumscribing circle in terms of the* sides *of the triangle.*

As in the last Article,

$$\text{Cot}\,\rho = \cot\frac{a}{2}\,.\,\cos PBE;$$

and as before,

$$\begin{aligned}
\text{Cos}\,PBE &= \cos(S'-A) = \cos\tfrac{1}{2}\{(B+C) - A\} \\
&= \cos\tfrac{1}{2}(B+C)\,.\,\cos\frac{A}{2} + \sin\tfrac{1}{2}(B+C)\,.\,\sin\frac{A}{2} \\
&= \frac{\sin\frac{A}{2}\,.\,\cos\frac{A}{2}}{\cos\frac{a}{2}}\,.\left(\cos\frac{b+c}{2} + \cos\frac{b-c}{2}\right).
\end{aligned}$$

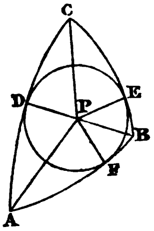
Gauss' Theorem, Art. 33, (ix.) and (viii.).

$$= \frac{\sin A}{\cos\frac{a}{2}}\,.\,\cos\frac{b}{2}\,.\,\cos\frac{c}{2};$$

$$\therefore\ \tan\rho = \frac{\tan\frac{a}{2}}{\cos PBE} = \frac{\sin\frac{a}{2}}{\cos\frac{b}{2}.\cos\frac{c}{2}.\sin A}$$

$$= \frac{2\sin\frac{a}{2}.\sin\frac{b}{2}.\sin\frac{c}{2}}{\sqrt{\{\sin S.\sin(S-a).\sin(S-b).\sin(S-c)\}}},\ \text{by (xix.)}$$

60. *To find the radius of a circle inscribed in a given triangle in terms of the* sides *of the triangle.*



Let ABC be the triangle; bisect $\angle A$ and $\angle C$ by AP and CP, arcs of great circles, which meet in P, and from P draw the arcs PD, PE, PF perpendicular to the sides.

Then

$\sin PD = \sin DCP.\sin CP = \sin PCE.\sin CP = \sin PE$;

$$\therefore\ PE = PD.\quad \text{Similarly, } PF = PD.$$

Therefore P is the pole of the inscribed circle.

$$\text{Let } PD = \rho'.\quad \text{Now } \sin CE = \tan\rho'.\cot\frac{C}{2};$$

$$\therefore\ \tan\rho' = \sin CE.\tan\frac{C}{2}.$$

And since the angles A, B, C are bisected, it is clear that $AD = AF$, $FB = BE$, $EC = CD$;

$$\therefore\ 2CE + 2AF + 2FB = a + b + c = 2S;$$

$$\therefore\ CE = S - (AF + FB) = S - c.$$

Also, by (xviii.), $\tan\frac{C}{2} = \sqrt{\frac{\sin(S-a).\sin(S-b)}{\sin S.\sin(S-c)}}$;

$$\therefore\ \tan\rho' = \sqrt{\frac{\sin(S-a).\sin(S-b).\sin(S-c)}{\sin S}}.$$

61. *To determine the radius of the inscribed circle in terms of the* angles *of the triangle.*

As in the last Article,

$$\tan\rho' = \sin CE \,.\, \tan\frac{C}{2},$$

and $\sin CE = \sin(S - c) = \sin\tfrac{1}{2}\{(a+b) - c\}$

$$= \sin\tfrac{1}{2}(a+b)\,.\,\cos\frac{c}{2} - \cos\tfrac{1}{2}(a+b)\,.\,\sin\frac{c}{2}$$

$$= \frac{\sin\frac{c}{2}\,.\,\cos\frac{c}{2}}{\sin\frac{C}{2}}\,.\,\left\{\cos\tfrac{1}{2}(A-B) - \cos\tfrac{1}{2}(A+B)\right\}$$

Gauss' Theorem, Art. 33, (xi.) and (ix.).

$$= \frac{\sin c}{\sin\frac{C}{2}}\,.\,\sin\frac{A}{2}\,.\,\sin\frac{B}{2};$$

$$\therefore\ \cot\rho' = \frac{\sin\frac{C}{2}}{\sin\frac{A}{2}\,.\,\sin\frac{B}{2}}\,.\,\frac{1}{\sin c}$$

$$= \frac{2\cos\frac{A}{2}\,.\,\cos\frac{B}{2}\,.\,\cos\frac{C}{2}}{\sqrt{\{-\cos S'\,.\,\cos(S'-A)\,.\,\cos(S'-B)\,.\,\cos(S'-C)\}}},\ \text{by (xxiii.)}.$$

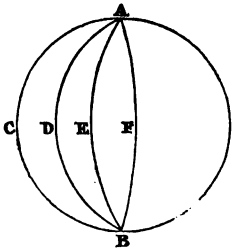
CHAPTER V.

ON THE AREAS OF SPHERICAL TRIANGLES, AND THE SOLUTION OF TRIANGLES WHOSE SIDES ARE SMALL COMPARED WITH THE RADIUS OF THE SPHERE.

62. Def. The portion of the surface of a sphere which is intercepted between two great semicircles, as *ACB* and *ADB*, is called *a lune*. (Fig. Art. 63.)

63. *To find the area of a lune.*

If *ACBD*, *ADBE*, *AEBF* be lunes having each the same angle at *A*, it is clear that they are all equal, and if the angle *CAD* be repeated any number of times, the area *ACBDA* will be repeated the same number of times;



i. e. the area of a lune varies as its angle;

$$\therefore \quad \frac{\text{area of lune whose angle is } A^0}{\text{area of a sphere (whose angle is } 360^0)} = \frac{A}{360};$$

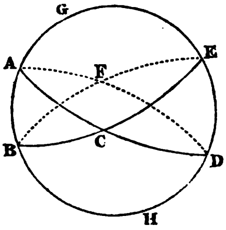
And area of a sphere $= 4\pi r^2$, if $r =$ radius of the sphere;

(Hymer's Integral Calculus, p. 153.)

$$\therefore \text{ area of lune whose angle is } A^0 = \frac{A}{360} \cdot 4\pi r^2$$

$$= \frac{A}{180} \cdot 2\pi r^2.$$

64. *To find the area of a spherical triangle.*



Let ABC be a triangle upon a hemisphere $ABDE$, Art. 14. Cor. 2., and let AC, BC, be produced till they meet again in F, which is on the side of the sphere turned from the spectator.

Then since CDF = a semicircle = ACD,

$$\therefore \ DF = CA.$$

similarly, $BC = EF$; and $\angle DFE = \angle ACB$;

$\therefore \ \triangle DFE = \triangle ACB$ in every respect.

Now Σ=area of $\triangle ABC$

$$=\text{surface of hemisphere} - BHDC - AGEC - DCE$$

$$=2\pi r^2 - (\text{lune}AHD - \Sigma) - (\text{lune}BGE - \Sigma) - (\text{lune}CDFE - \Sigma)$$

$$=2\pi r^2\left(1 - \frac{A}{180} - \frac{B}{180} - \frac{C}{180}\right) + 3\Sigma; \quad \text{by Art. 63.}$$

$$\therefore \ \Sigma = (A + B + C - 180^0)\cdot\frac{\pi r^2}{180^0}$$

$$= \frac{E}{180}\cdot\pi r^2; \qquad \text{if } E = A + B + C - 180^0.$$

Def. The quantity $A + B + C - 180^0$ by which the sum of the degrees in the angles of the spherical triangle exceeds 180^0, is called *the spherical excess* of the triangle.

Cor. 1. Hence for all triangles described on the same sphere,

$$\Sigma \propto A + B + C - 180^0.$$

And on this account $A + B + C - 180^0$ has sometimes been taken as the measure of the surface of a triangle.

COR. 2. *To find the area of a spherical polygon.*

Divide the polygon into as many triangles as it has sides, by means of arcs of great circles drawn from each of the angular points to any point within the polygon. Let n be the number of sides of the polygon.

Then Area = area of the n triangles

$$= \frac{\pi r^2}{180^0} \cdot (\text{angles of the triangles} - n \cdot 180^0)$$

$$= \frac{\pi r^2}{180^0} \cdot \{\text{angles of polygon} - (n-2)\,180^0\}.$$

65. CAGNOLI'S THEOREM. *To shew that if* E *be the spherical excess,*

$$\textit{then } \sin\frac{E}{2} = \frac{\sqrt{\{\sin S \cdot \sin(S-a) \cdot \sin(S-b) \cdot \sin(S-c)\}}}{2\cos\frac{a}{2} \cdot \cos\frac{b}{2} \cdot \cos\frac{c}{2}}.$$

$$\text{Sin}\frac{E}{2} = \sin\tfrac{1}{2}(A+B+C-180) = \sin\left\{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(180^0 - C)\right\}$$

$$= \sin\tfrac{1}{2}(A+B) \cdot \sin\frac{C}{2} - \cos\tfrac{1}{2}(A+B) \cdot \cos\frac{C}{2}.$$

$$\text{And by} \begin{cases} (\text{viii.}),\ \sin\tfrac{1}{2}(A+B) = \dfrac{\cos\frac{C}{2}}{\cos\frac{c}{2}} \cdot \cos\tfrac{1}{2}(a-b) \\[2ex] (\text{ix.}),\ \cos\tfrac{1}{2}(A+B) = \dfrac{\sin\frac{C}{2}}{\cos\frac{c}{2}} \cdot \cos\tfrac{1}{2}(a+b). \end{cases}$$

$$\therefore\ \sin\frac{E}{2} = \left\{\cos\tfrac{1}{2}(a-b) - \cos\tfrac{1}{2}(a+b)\right\} \cdot \frac{\sin\frac{C}{2} \cdot \cos\frac{C}{2}}{\cos\frac{c}{2}}$$

$$= \frac{\sin\frac{a}{2} \cdot \sin\frac{b}{2}}{\cos\frac{c}{2}} \cdot \sin C.$$

$$= \frac{\sin\frac{a}{2} . \sin\frac{b}{2}}{\cos\frac{c}{2}} . \frac{2}{\sin a . \sin b} . \sqrt{\{\sin S . \sin(S-a) . \sin(S-b) . \sin(S-c)\}}$$

$$= \frac{\sqrt{\{\sin S . \sin(S-a) . \sin(S-b) . \sin(S-c)\}}}{2\cos\frac{a}{2} . \cos\frac{b}{2} . \cos\frac{c}{2}} .$$

66. LLHUILLIER'S THEOREM. *To shew that*

$$\tan\frac{E}{4} = \sqrt{\{\tan\tfrac{1}{2}S . \tan\tfrac{1}{2}(S-a) . \tan\tfrac{1}{2}(S-b) . \tan\tfrac{1}{2}(S-c)\}}.$$

By Pl. Trig. Art. 51, (2) and (3),

$$\frac{\sin\frac{1}{2}(A+B) - \sin\frac{1}{2}(180^\circ - C)}{\cos\frac{1}{2}(A+B) + \cos\frac{1}{2}(180^\circ - C)} = \frac{\sin\frac{1}{4}(A+B+C-180^\circ)}{\cos\frac{1}{4}(A+B+C-180^\circ)}$$

$$= \tan\frac{E}{4};$$

$$\therefore \tan\frac{E}{4} = \frac{\sin\frac{1}{2}(A+B) - \cos\frac{C}{2}}{\cos\frac{1}{2}(A+B) + \sin\frac{C}{2}}$$

$$= \frac{\cos\frac{1}{2}(a-b) - \cos\frac{c}{2}}{\cos\frac{1}{2}(a+b) + \cos\frac{c}{2}} . \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}},$$

Gauss' Theorem, Art. 33. (viii) and (ix).

$$= \frac{\sin\frac{1}{4}(c+a-b) . \sin\frac{1}{4}(c+b-a)}{\cos\frac{1}{4}(a+b+c) . \cos\frac{1}{4}(a+b-c)} . \sqrt{\frac{\sin S . \sin(S-c)}{\sin(S-a) . \sin(S-b)}}$$

By (xvi) and (xvii).

$$= \sqrt{\{\tan\tfrac{1}{2}S . \tan\tfrac{1}{2}(S-a) . \tan\tfrac{1}{2}(S-b) . \tan\tfrac{1}{2}(S-c)\}};$$

by expressing the sines under the root in terms of the sines and cosines of half the angles.

67. There are some other values of E, the Spherical Excess, which are symmetrical functions of the sides of the triangle; but as these are not of any real use, since they are not adapted to logarithmic computation, we shall not here give the demonstrations of them. The expressions will be found among the Examples which are given at the end of the book,—and may be proved without much trouble by Gauss' formulæ.

68. In the following Articles we shall suppose x, y, z to be the lengths of the arcs opposite to the angles A, B, C of the spherical triangle ABC which is described on a sphere of radius r, and whose sides are small compared with the radius of the sphere. Also, A', B', C' are the angles of the *plane* triangle whose sides are x, y, z; and A_1, B_1, C_1 are the angles which are observed for the angles of ABC, the errors α, β, γ being made at the respective observations; so that

$$A = A_1 + \alpha, \quad B = B_1 + \beta, \quad C = C_1 + \gamma.$$

69. *The area of a triangle whose sides are small compared with the radius of the sphere being approximately known, required the number of seconds in its spherical excess.*

Let the triangle be described on the surface of the Earth, which we will suppose to be a sphere.

Let r = number of linear feet in the Earth's radius,

n = number of square feet in the area of the triangle

$$= \frac{E}{180} \cdot \pi r^2. \quad \text{Art. 64};$$

$$\therefore \; n = E \,.\, 60 \,.\, 60 \,.\, \frac{\pi r^2}{180 \,.\, 60 \,.\, 60}.$$

Now $\dfrac{\text{length of an arc of } 1^\circ}{\text{radius}} = \dfrac{1}{180}\,.\,\pi$. Pl. Trig. Chap. IV.

$$\therefore\ \frac{\pi r}{180} = \text{length of } 1^\circ\text{, in feet,}$$

$$= 60859{\cdot}1 \times 6\text{, by actual measurement;}$$

$$\therefore\ r = \frac{180}{\pi}\,.\,60859{\cdot}1 \times 6;$$

$$\therefore\ n = E\,.\,60\,.\,60\,.\,\frac{180}{\pi\,.\,60\,.\,60}\,.\,(60859{\cdot}1 \times 6)^2$$

$$= E\,.\,60\,.\,60\,.\,\frac{180\,.\,(6085{\cdot}91)^2}{3{\cdot}141592};$$

$$\therefore\ l_{10}(E\,.\,60\,.\,60) = l_{10}n - (l_{10}180 + 2l_{10}6085{\cdot}91 - l_{10}3{\cdot}141592)$$

$$= l_{10}n - 9{\cdot}3267736 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

where $E\,.\,60\,.\,60$ is the number of seconds in the spherical excess.

NOTE. This reasoning being general, it is clear that the number of seconds in the spherical excess may always be determined from

$$n = (E\,.\,60\,.\,60)\,.\,\frac{\pi r^2}{180\,.\,60\,.\,60},$$

if the radius of the sphere be known, *whatever* be its magnitude.

COR. We may determine n in the following manner.

n = area of the spherical triangle, in square feet.

= area of the plane triangle whose sides are x, y, z, and whose angles are the observed angles A_1, B_1, C_1,—nearly.

$= \frac{1}{2}y\,.\,z\,.\,\sin A_1$, or $= \frac{1}{2}x^2 \dfrac{\sin B_1\,.\,\sin C_1}{\sin(B_1 + C_1)}$*. Pl. Trig. Chap. VI.

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[*] General Roy in the Trigonometrical Survey of England approximately determined the area of the triangle to a sufficient degree of accuracy by laying down on paper the base and the observed angles at the base, and measuring the perpendicular from the vertex to the base by the compasses on a scale.

70. Ex. *If the observed angles and a side* x *of the triangle* ABC *be*

$$\left.\begin{aligned} A_1 &= 42^0,\ 2',32'' \\ B_1 &= 67^0,55',39'' \\ C_1 &= 70^0,\ 1',48'' \end{aligned}\right\} = 179^0,59',59''.$$

$$x = 27404{\cdot}2 \text{ feet};$$

required the number of seconds in the sum of the errors made in observing the angles.

Here the *apparent* spherical excess is

$$179^0, 59', 59'' - 180^0 = -1''.$$

Now x being the side opposite to the angle A_1,

$$n = \tfrac{1}{2} \cdot x^2 \cdot \frac{\sin B_1 \cdot \sin C_1}{\sin (B_1 + C_1)},$$

and $l_{10}(E \cdot 60 \cdot 60) = l_{10}\left\{\tfrac{1}{2} \cdot x^2 \cdot \dfrac{\sin B_1 \cdot \sin C_1}{\sin (B_1 + C_1)}\right\} - 9{\cdot}3267736$

$$= l_{10}\left\{\tfrac{1}{2} \cdot (27404{\cdot}2)^2 \cdot \frac{\sin 67^0,\ 55',\ 39'' \times \sin 70^0,\ 1',\ 48''}{\sin 137^0, 57', 27''}\right\} - 9{\cdot}3267736.$$

Add, $2 \times l_{10} 27404{\cdot}2 \qquad = 2 \times 4{\cdot}4378172 = 8{\cdot}8756344$

$L \sin 67^0,\ 55',\ 39'' \qquad = 9{\cdot}9669434$

$L \sin 70^0,\ \ 1',\ 48'' \qquad = 9{\cdot}9730685$

$28{\cdot}8156463$

Subtract $l_{10} 2 \qquad = \cdot3010300$

$L \sin 137^0, 57', 27''$,

or $L \sin\ 42^0,\ 2', 33'' = 9{\cdot}8258684$

10

$9{\cdot}3267736$

29.4536720

$\bar{1}{\cdot}3619743 = l_{10}(\cdot23)$,

nearly.

And therefore the *computed* spherical excess, which may be supposed to be the *real* spherical excess, is ·23″.

Hence it appears that the whole error of observation, viz. *real* spherical excess – *apparent* spherical excess, is ·23″ – (–1″), or 1″·23, which has to be added to the three observed angles A_1, B_1, C_1, in such proportions as the observer's judgment may direct. (See the next Article.)

71. *To shew how the observed angles of a spherical triangle whose sides are small compared with the radius of the sphere may be freed from the errors of observation.*

Let A, B, C, be the *real* angles of the triangle,

A_1, B_1, C_1, the *observed* angles,

α, β, γ the errors made in observing A, B, C, respectively.

$$\text{So that} \quad A + B + C = (A_1 + \alpha) + (B_1 + \beta) + (C_1 + \gamma);$$

$$\therefore \ (A + B + C - 180^0) = (A_1 + B_1 + C_1 - 180^0) + (\alpha + \beta + \gamma);$$

$$\therefore \ \alpha + \beta + \gamma = (A + B + C - 180^0) - (A_1 + B_1 + C_1 - 180^0)$$

$$= \text{real spherical excess} - \text{computed spherical excess}.$$

Now if, by the method of the last Article, we approximate closely to the real spherical excess, we can find by this equation the *sum* of the errors of observation which have been made. The *distribution* however of this sum,—that is, the determining what part of the whole error is to be assigned to each angle individually,—must evidently be left to the judgment of the observer, who, from knowing the state of the atmosphere at the times of observation, may thence form an opinion how far the observations can be depended on, and may then to each of the observed angles assign such a portion of the whole error as he thinks most likely to lead to a correct solution of the triangle.

If by the method of Art. 69, $\alpha + \beta + \gamma$, the sum of the errors of observation, be found, and thence the angles A, B, C, be determined by the arbitrary assignment of the several parts of this whole error to each of the observed angles A_1, B_1, C_1, the *sum* of the errors of observation may be supposed to be got rid of. Yet it is highly probable that the judgment of the observer has not been absolutely correct in this arbitrary assignment of the parts of the whole error. The following theorem will point out what relation the sides of the triangle ought to bear to each other, in order that the small quantities by which the *corrected* angles differ from the *real* angles of the triangle, may have the least possible effect in producing errors in the determination of the other two sides of the triangle from a measured side and the *corrected* angles.

72. *Having given the corrected angles and one side of a spherical triangle whose sides are small compared with the radius of the sphere, required the relation which the sides of the triangle ought to bear to each other, in order that the other sides may be determined from these data with the least probable amount of error.*

Let x, y, z be the sides opposite to the *real* angles of the triangle A, B, C; α', β', γ', the errors of the *corrected* angles; therefore $A + \alpha'$, $B + \beta'$, $C + \gamma'$ are the *corrected* angles.

Then since the spherical excess, and therefore the *sum* of the real angles of the triangle, is supposed to be known *exactly*, the sum of the corrected angles is known exactly, and therefore the sum of their errors α', β', γ' must vanish;

$$\therefore\ \alpha' + \beta' + \gamma' = 0, \quad \text{or } \alpha' = -(\beta' + \gamma').$$

(We shall throughout this Article neglect powers of α', β', γ' of any order above the first.)

Now, considering the spherical triangle as very nearly a plane triangle,

$$z = x \cdot \frac{\sin(C + \gamma')}{\sin(A + \alpha')}$$

$$= x \cdot \frac{\sin(C + \gamma')}{\sin\{A - (\beta' + \gamma')\}}$$

$$= x \cdot \frac{\sin C \cdot \cos\gamma' + \cos C \cdot \sin\gamma'}{\sin A \cdot \cos(\beta' + \gamma') - \cos A \cdot \sin(\beta' + \gamma')}$$

$$= x \cdot \frac{\sin C}{\sin A} \cdot \frac{1 + \gamma' \cdot \cot C}{1 - (\beta' + \gamma') \cdot \cot A}, \quad \text{nearly.}$$

$$= x \cdot \frac{\sin C}{\sin A} \cdot \{1 + \gamma' \cdot \cot C\} \cdot \{1 + (\beta' + \gamma') \cdot \cot A\}$$

$$= x \cdot \frac{\sin C}{\sin A} \cdot \{1 + \gamma' \cdot (\cot A + \cot C) + \beta' \cdot \cot A\}$$

$$= x \cdot \frac{\sin C}{\sin A} + x \cdot \left\{\gamma' \cdot \frac{\sin C}{\sin A} \cdot \frac{\sin(A + C)}{\sin A \cdot \sin C} + \beta' \cdot \frac{\cos A \cdot \sin C}{(\sin A)^2}\right\}.$$

Now, $\sin B = \sin(\pi - B) = \sin(A + C)$, nearly;

Therefore the error in the value of z, or $z - x\dfrac{\sin C}{\sin A}$, is

$$x \cdot \left\{\gamma' \cdot \frac{\sin B}{(\sin A)^2} + \beta' \cdot \frac{\cos A \cdot \sin C}{(\sin A)^2}\right\} \ldots\ldots\ldots\ldots (1).$$

Similarly the error of y is

$$x \cdot \left\{\beta' \cdot \frac{\sin C}{(\sin A)^2} + \gamma' \cdot \frac{\cos A \cdot \sin B}{(\sin A)^2}\right\} \ldots\ldots\ldots\ldots (2).$$

The question now reduces itself to this, viz. to determine what values of A, B, C, make both the quantities (1) and (2) the least possible.

Since $\alpha' + \beta' + \gamma' = 0$, two of these quantities must be of the same sign and one of the other sign. Wherefore the *probability* is that any particular two, as β' and γ', are of *different* signs. If β' and γ' *be* of different signs, the expressions (1) and (2) are diminished in magnitude by giving $\cos A$ a *positive* value*, that is, by supposing A to be less than a right angle.

If we make this further supposition, that the errors β' and γ' though different in sign are yet nearly equal in magnitude, it is clear that (1) and (2) satisfy this hypothesis, if B be nearly equal to C.

The conclusion therefore we arrive at is, *The greatest probability* of correctness in solving a small spherical triangle from three observed angles and a measured side, is when the angle opposite to the known side is less than a right angle, and the other two sides are nearly equal. And these conditions will be best fulfilled, for a *series* of triangles, if care be taken that *each* triangle be nearly equilateral.

73. It may be as well to recount the *gratuitous* suppositions which have been made in the last Article.

I. The spherical excess has been accurately determined.

II. The errors β' and γ' are of different signs.

III. These errors are of the same magnitude.

For any particular triangle it is very probable that some of these suppositions may not be true. It may happen that none of them may be correct. In very few cases indeed will they *all* be fulfilled.

With respect to the Spherical Excess, we may generally suppose it to be accurately known.

With respect to the signs of β' and γ', since two of the quantities a', β', γ', are of the same, and the third is the contrary sign, the *probability* of β' and γ' being of *different* signs, is twice as great as the probability that they are of the same sign. And if they *be* of different signs, then (1) and (2) of Art. 72. shew that the errors will not be so great if $\cos A$ be positive, as if it be negative. This consideration renders it advisable that each of the angles of the triangle should be less than a right angle,—which will be the case, if the triangle be nearly equilateral.

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* It is evidently more advantageous in determining a series of triangles from one another, that the errors should be inconsiderable and equally diffused through all, than that any one calculated side should by differing much from its real value, affect with great errors all the triangles successively determined from it. If, β' and γ' being of different signs, $\cos A$ become *negative*, the errors (1) and (2) are increased, and considerable errors may be thus introduced into the calculations.

But if the third hypothesis be admitted, since a' vanishes in this case, the angle A is supposed, (which is highly improbable), to have been determined with mathematical exactness. Besides, it is very unlikely indeed that an observer after making an error $+\beta'$ in observing an angle, should make an error $-\beta'$, (for $\gamma' = -\beta'$), in observing, with the same instrument, an angle C which is nearly of the same magnitude as B.

74. LEGENDRE'S THEOREM.

If each of the angles of a spherical triangle whose sides are small when compared with the radius of the sphere be diminished by one third of the spherical excess, the triangle may be solved as a plane triangle whose sides are equal to the sides of the spherical triangle, and whose angles are these reduced angles.

Let x, y, z be the lengths of the sides respectively opposite to the angles A, B, C of a small *spherical* triangle described on the sphere whose radius is r. Let A', B', C' be the angles of that *plane* triangle whose sides are x, y, z.

$$\text{Now} \quad \cos A' = \frac{y^2 + z^2 - x^2}{2yz},$$

$$\text{and} \quad \cos A \,.\, \sin b \,.\, \sin c = \cos a - \cos b \,.\, \cos c,$$

$$\text{or} \quad \cos A \,.\, \sin\frac{y}{r} \,.\, \sin\frac{z}{r} = \cos\frac{x}{r} - \cos\frac{y}{r} \,.\, \cos\frac{z}{r}.$$

Expanding the sines and cosines of these small angles in terms of the angles, and neglecting powers above the fourth, as we shall do throughout this investigation,—(there will therefore be no powers or products of x, y, z of more than four dimensions),—we have

$$\cos A = \frac{\left(1 - \frac{x^2}{2r^2} + \frac{x^4}{2.3.4r^4}\right) - \left(1 - \frac{y^2}{2r^2} + \frac{y^4}{2.3.4r^4}\right) \cdot \left(1 - \frac{z^2}{2r^2} + \frac{z^4}{2.3.4r^4}\right)}{\left(\frac{y}{r} - \frac{y^3}{2.3r^3}\right) \cdot \left(\frac{z}{r} - \frac{z^3}{2.3r^3}\right)}$$

$$= \frac{\frac{1}{2r^2}(y^2 + z^2 - x^2) - \frac{y^2 z^2}{4r^4} + \frac{1}{24r^4} \cdot (x^4 - y^4 - z^4)}{\frac{yz}{r^2} \cdot \left(1 - \frac{y^2 + z^2}{6r^2}\right)}$$

$$= \frac{1}{2yz}\left\{(y^2 + z^2 - x^2) + \frac{1}{2r^2}(x^4 - y^4 - z^4 - 6y^2 z^2)\right\} \cdot \left\{1 + \frac{y^2 + z^2}{6r^2}\right\}$$

$$= \frac{y^2 + z^2 - x^2}{2yz} + \frac{1}{24yzr^2} \cdot \left\{\begin{array}{c} 2y^4 + 4y^2 z^2 + 2z^4 - 2x^2 y^2 - 2x^2 z^2 \\ + x^4 - y^4 - z^4 - 6y^2 z^2 \end{array}\right\}$$

$$= \cos A' - \frac{1}{24yzr^2} \cdot \{2x^2 y^2 + 2x^2 z^2 + 2y^2 z^2 - x^4 - y^4 - z^4\}$$

$$= \cos A' - \frac{(2yz)^2}{24yzr^2} \cdot \left\{1 - \left(\frac{y^2 + z^2 - x^2}{2yz}\right)^2\right\}$$

$$= \cos A' - \frac{yz}{6r^2} \cdot \{1 - (\cos A')^2\} \text{ ;}$$

$$\therefore \frac{yz}{6r^2} \cdot (\sin A')^2 = \cos A' - \cos A = \sin \tfrac{1}{2}(A - A') \cdot \sin \tfrac{1}{2}(A + A')$$

[Now A and A' are in practice expressed in degrees, &c.; when expressed by the circular measure, $\sin \frac{1}{2}(A - A')^\circ$ becomes

$$= \sin \tfrac{1}{2}(A - A')\frac{\pi}{180} = \tfrac{1}{2}(A - A')\frac{\pi}{180}, \text{ nearly, since the}$$

angle is very small.

Also $\sin\frac{1}{2}(A+A') = \sin A'$, nearly;

and $yz \,.\, \sin A' = 2\triangle A'B'C' = 2\triangle ABC$, nearly, $= 2 \,.\, \dfrac{E}{180} \,.\, \pi r^2$

Making these substitutions our equation becomes,]

$$\frac{2 \,.\, \dfrac{E}{180} \,.\, \pi r^2}{6r^2} \,.\, \sin A' = 2 \,.\, \tfrac{1}{2}(A-A') \,.\, \frac{\pi}{180} \,.\, \sin A';$$

$$\therefore \quad \frac{E}{3} = A - A';$$

$$\therefore \quad A' = A - \frac{E}{3}.$$

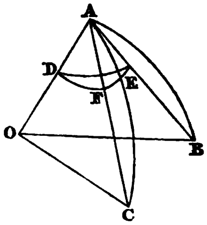
So $\quad B' = B - \dfrac{E}{3}$, and $C' = C - \dfrac{E}{3}$.

Hence if BC, or x, be measured, and one third of the computed spherical excess be subtracted from each of the observed angles A, B, C, we have the case of a *plane* triangle $A'B'C'$, whose angles $A - \dfrac{E}{3}$, $B - \dfrac{E}{3}$, $C - \dfrac{E}{3}$ are known, and the side x being given, AB and AC can be found. And AB and AC are y and z, the other arcs which are the sides of the spherical triangle ABC.

NOTE. It may here be remarked that the determination of E is a matter of considerable importance, when triangles have to be solved whose sides are small compared with the radius of the sphere on which they are described. We have already seen the use this quantity is of, in correcting the observed angles, Art. 71, and here we find it absolutely necessary, if the triangle be solved by Legendre's approximate method. How it may be determined from three measured or observed parts of the triangle is given in Art. 69, Cor.

75. *To find the angle contained between the chords of two spherical arcs which subtend given angles at the centre of the sphere, the angle between the arcs themselves being also given.*

Let AB and AC be the arcs, O the centre of the sphere. Let the lines AO, AB, AC meet the surface of a sphere, which is described with centre A and any radius AD, in the points D, E, F respectively. Then the angle EDF is the inclination of the planes BAO and CAO, i.e. the angle contained between the arcs AB and AC, or $\angle A$.



Let $\dfrac{AB}{AO} = c$, $\dfrac{AC}{AO} = b$.

Now $\angle FAO = \frac{1}{2}(\pi - \angle AOC) = \frac{1}{2}(\pi - b)$.

So $\angle EAO = \frac{1}{2}(\pi - c)$.

And from the triangle EDF,

$$\text{Cos}\, EF = \cos DF \,.\, \cos DE + \sin DF \,.\, \sin DE \,.\, \cos EDF;$$

$$\text{Or} \quad \cos EAF = \sin\frac{b}{2} \,.\, \sin\frac{c}{2} + \cos\frac{b}{2} \,.\, \cos\frac{c}{2} \,.\, \cos A \;\ldots\ldots\ldots\ldots \;(1).$$

76. We will now deduce from the result obtained in the last Article, two formulæ which are convenient for determining practically the angle EAF.

Let $\angle EAF = A - \theta$;

Then

$$\cos(A - \theta) - \cos A = \sin\frac{b}{2} \,.\, \sin\frac{c}{2} + \left(\cos\frac{b}{2} \,.\, \cos\frac{c}{2} - 1\right) . \cos A;$$

And, Pl. Trig. Art. 54,

$$\left\{\begin{array}{l} \sin\dfrac{b}{2} \,.\, \sin\dfrac{c}{2} = \left(\sin\dfrac{b+c}{4}\right)^2 - \left(\sin\dfrac{b-c}{4}\right)^2, \\ \cos\dfrac{b}{2} \,.\, \cos\dfrac{c}{2} = \left(\cos\dfrac{b+c}{4}\right)^2 - \left(\sin\dfrac{b-c}{4}\right)^2 = 1 - \left(\sin\dfrac{b+c}{4}\right)^2 - \left(\sin\dfrac{b-c}{4}\right)^2; \end{array}\right.$$

Therefore our equation becomes

$$2\sin\left(A-\frac{\theta}{2}\right).\sin\frac{\theta}{2}=\left\{\left(\sin\frac{b+c}{4}\right)^2-\left(\sin\frac{b-c}{4}\right)^2\right\}$$

$$-\left\{\left(\sin\frac{b+c}{4}\right)^2+\left(\sin\frac{b-c}{4}\right)^2\right\}.\cos A$$

$$=\left(\sin\frac{b+c}{4}\right)^2.(1-\cos A)-\left(\sin\frac{b-c}{4}\right)^2.(1+\cos A)\ \ldots\ldots\ldots\ldots\ (1).$$

For an approximation to the value of θ, make $\sin\frac{\theta}{2}=\frac{\theta}{2}$, and $\sin\left(A-\frac{\theta}{2}\right)=\sin A$;

Then, since $\frac{1-\cos A}{\sin A}=\frac{2\left(\sin\frac{A}{2}\right)^2}{2.\sin\frac{A}{2}.\cos\frac{A}{2}}=\tan\frac{A}{2}$,

and $\frac{1+\cos A}{\sin A}=\cot\frac{A}{2}$;

$$\therefore\ \theta=\left(\sin\frac{b+c}{4}\right)^2.\tan\frac{A}{2}-\left(\sin\frac{b-c}{4}\right)^2.\cot\frac{A}{2};$$

Therefore the number of seconds in θ

$$=\frac{\theta}{\sin 1''}=\frac{1}{\sin 1''}.\left(\sin\frac{b+c}{4}\right)^2.\tan\frac{A}{2}-\frac{1}{\sin 1''}.\left(\sin\frac{b-c}{4}\right)^2.\cot\frac{A}{2};$$

And determining the two terms of the second member of this equation separately by means of tables of logarithms, we obtain the number of seconds in θ, and thence may determine the angle $A-\theta$, which is contained by the straight lines BA and CA.

Cor. If y, z, be the measured lengths of the arcs AB, AC,

$$\sin\frac{b \pm c}{4} = \sin\frac{y \pm z}{4r} = \frac{y \pm z}{4r}, \text{ nearly.}$$

$\therefore$ number of seconds in θ

$$= \frac{1}{\sin 1''} \cdot \left\{ \left(\frac{y+z}{4r}\right)^2 . \tan\frac{A}{2} - \left(\frac{y-z}{4r}\right)^2 . \cot\frac{A}{2} \right\}.$$

If then the radius of the Earth and the sides of the spherical triangle be approximately known, the chordal triangle can be determined, since its sides can be found by the expression, chord $= 2r . \sin\left(\frac{1}{2}\frac{\text{arc}}{\text{rad.}}\right)$, and its angles can be determined by one of the last two Articles.

77. Another method is to solve the triangle by the rules laid down for the solution of spherical triangles whose sides are *not* small in comparison with the radius of the sphere. In this case the logarithms of the sines and tangents of the sides, which are very small, must be found by the methods pointed out in Appendix iii. to Pl. Trig.

CHAPTER VI.

ON GEODETIC MEASUREMENTS.

78. GEODETIC MEASUREMENTS are employed for obtaining a correct representation of a part of the Earth's surface, which is too large in extent to be considered as lying in one plane.

A horizontal line of considerable length is first measured, which is called a *base*.

Next, an object is fixed upon, so situated that it forms with the extremities of the base a triangle which is nearly equilateral, Arts. 72, 73. The angles of this triangle are then measured by a theodolite,—an instrument which will be described hereafter, Art. 85,—and the remaining parts of the triangle computed by some one of the following methods:

1. By the common processes of Spherical Trigonometry, Chap. v.

2. By the chordal triangle, Arts. 75, 76.

3. By diminishing each of the observed angles by one third of the spherical excess, and then treating the figure as a plane triangle*. Art. 74.

79. *The use of Geodetic Measurements.*

The form and position of the polygon being thus determined, it may be represented on paper according to any projection of the sphere, (Hymers' Astronomy, Appendix I.), and a map of a country may be thus obtained. If the object in view be to determine the figure and dimensions of the Earth with great accuracy, the length of an arc *mn* passing through two given points of the polygon may be determined by calculation. If these points, *m* and *n*, be situated on the same meridian, and the difference of the zenith distances of the same fixed star when on the meridian be noted by observers at *m* and *n*, the Earth's radius of curvature at the middle point of the calculated arc may be approximately found. From arcs thus measured in different latitudes the figure of the Earth has been determined with great exactness. (Astronomy, Chap. II. Arts. 91. 98. 102. 104.)*

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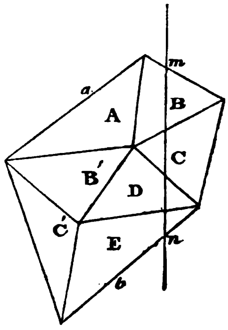
* The first of these methods was preferred by Delambre, the second was used in the English, and the third in the French, survey.



Next, one of the computed sides is taken as the base of another triangle, whose angles are observed, and sides computed; and thus by a series of triangles, the figure and dimensions of a polygonal area on the Earth's surface are determined. The Earth is, in fact, a spheroid, but it differs so little from a sphere, that *each* triangle may be supposed, without appreciable error, to be described on the *same* sphere. If the triangulation be carried over a very extensive tract of country, it will become necessary to take into consideration the alteration which a change of latitude produces in the Earth's radius.

80. *Base of Verification.*

A side of the last triangle of the series, after being thus computed, is carefully measured. The degree of exactness with which the measured coincides with the computed length tests the accuracy of the survey. Any considerable error is easily detected in the course of the calculations, in the following manner. (Fig. Art. 79.) If the value of *b*, as calculated from the original base a through the triangles A, B, C, D, E, be found to agree closely with its value as computed from the same base through a different set of triangles A, B', C', E, it may be presumed to have been accurately determined*.

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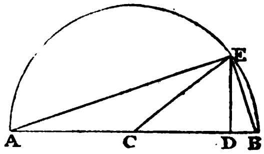
* The student will find this part of the subject clearly and concisely treated of in the Articles on "Trigonometry" and "the Figure of the Earth," written by the Astronomer Royal for the Encyclopædia Metropolitana. From Section 179 of the former of these Articles, 72 of this treatise has been taken.

For a more particular description of the details of Geodetic operations, Lardner's Trigonometry may be consulted, or any of the published accounts of the English survey.

81. *Corrections of measurements.*—There are several causes which are productive of error in Geodetic Measurements. The refraction of light is affected by the perpetual variations of the atmosphere in density and temperature, and thus the observed angles cannot always be relied on. It is almost impossible to find a line perfectly straight and perfectly horizontal, to measure for a base. Neither can a portion of the Earth be found which is altogether free from inequalities of surface. To get rid of the errors which are thus introduced, several corrections are used. Some of these, which do not depend on experiment, but are capable of mathematical investigation, will now be given.

(1). *If there be a* slight *rise in the line of the base, to determine the reduction to the horizon.*



Let CB be horizontal, CE the line of the base, ED the small rise,—which is obtained by levelling with a spirit-level, (Art. 84.) Describe a circle with centre C and radius CE.

Then $BD = CB - CD = CE - CD =$ the Reduction to the horizon.

$$\text{And}\quad BD = \frac{BE^2}{AB} = \frac{BE^2}{2CE} = \frac{DE^2}{2CE}\dagger,\ \text{nearly};$$

the formula made use of in the English survey of 1786, &c.

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* In the English trigonometrical survey of 1787 and succeeding years, the original base on Hampstead Heath was by admeasurement 27404·2 feet; and the base of verification on Salisbury Plane was, as measured, 36575·401 feet, and as computed through three different series of triangles,

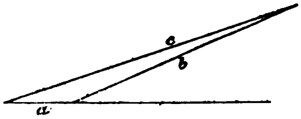
36574·3
36574·6
36574·9,

any of which approximations is sufficiently near for all practical purposes.

† If $BD = x$, $CE = a$, $DE = b$, $x = \frac{BE^2}{2a} = \frac{b^2 + x^2}{2a}$,

and neglecting powers of x above the first, $x = \frac{b^2}{2a}$.

(2). *Let the measured base, instead of being* one *straight line, consist of* two *straight lines,* a *and* b, *enclosing an angle* $\pi - \theta$, *where* θ *is very small. Required the correction to find* c.



$$c^2 = a^2 + b^2 + 2ab\cos\theta$$

$$= a^2 + b^2 + 2ab \cdot \left(1 - \frac{\theta^2}{2}\right)$$

$$= (a+b)^2 \cdot \left\{1 - \frac{ab}{(a+b)^2} \cdot \theta^2\right\};$$

$$\therefore\ c = (a+b) \cdot \left\{1 - \frac{ab}{2(a+b)^2} \cdot \theta^2\right\}, \quad \text{nearly};$$

$$= (a+b) - \frac{ab}{2(a+b)} \cdot \theta^2.$$

$$\therefore\ \text{Correction} = (a+b) - c = \frac{ab}{2(a+b)} \cdot \theta^2.$$

Or if θ be an angle of n'', where n is very small,

$$\text{Correction, in seconds,} = \frac{1}{\sin 1''} \cdot \frac{ab \cdot (n \sin 1'')^2}{2(a+b)} = \frac{abn^2 \cdot \sin 1''}{2(a+b)}.$$

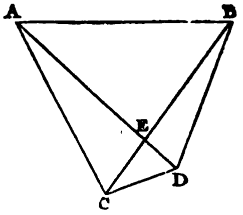
[It is seldom necessary, in practice, to apply this correction.]

(3). *From observations made at a point* D, *which is at a small known distance from a signal* C, *required to find the angle which* A *and* B *subtend at* C.

Let $CD = a$, and the observed angle $ADB = A$,

$\alpha = \angle ADC$, the angle subtended at D by the left hand object and the signal C.

$AC = r$, $BC = l$, the distances of the right and left hand objects from C;—which are approximately known from AB, and the observed angles BAC, ABC.



Let $\angle BCA$, the required angle, $= \theta$.

Then $\angle A + \angle DBE =$ exterior angle BEA

$$= \theta + \angle CAD;$$

$\therefore\ \theta - A$, the reduction, $= \angle DBE - \angle CAD$

$$= \sin DBE - \sin CAD, \text{ nearly},$$

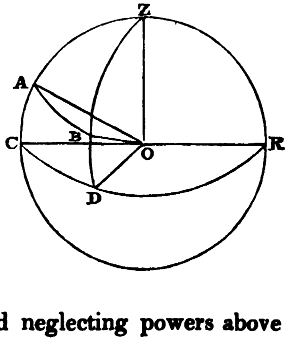
$$= \frac{a}{r}.\sin CDB - \frac{a}{l}.\sin ADC = a.\left\{\frac{\sin(A+a)}{r} - \frac{\sin a}{l}\right\}$$

or $$= \frac{a}{\sin 1''}.\left\{\frac{\sin(A+a)}{r} - \frac{\sin a}{l}\right\},$$

when expressed in seconds.

(4). *The altitudes or depressions, (which are small quantities), of two objects having been observed, and also the angle they subtend at a certain point, to find the horizontal angle they subtend.*—(Encycl. Metrop.)

Let Z be the Observer's zenith, $CDRO$ his horizon, A and B the two objects; $\angle BOA = \theta$; $\angle DOC$, its reduction to the horizon, $= \theta + x$; $CA = h$, $DB = h'$.



From the ΔAZB,

$$\operatorname{Cos} CZD = \frac{\cos\theta - \cos ZA.\cos ZB}{\sin ZA.\sin ZB}$$

$$= \frac{\cos\theta - \sin h.\sin h'}{\cos h.\cos h'}$$

[Making $\sin h = h$, $\cos h = 1 - \frac{h^2}{2}$, and neglecting powers above the second, this becomes,]

$$=\frac{\cos\theta - hh'}{1-\dfrac{h^2+h'^2}{2}}$$

$$=(\cos\theta - hh').\left(1+\frac{h^2+h'^2}{2}\right),\quad \text{nearly},$$

$$=\cos\theta - hh' + \tfrac{1}{2}(h^2+h'^2).\cos\theta.$$

But $\text{Cos}\, CZD = \cos(\theta + x) = \cos\theta.\cos x - \sin\theta.\sin x.$

$$=\cos\theta - x.\sin\theta,\quad \text{nearly}.$$

And from these two equations,

$$x=\frac{hh'}{\sin\theta}-\tfrac{1}{2}(h^2+h'^2).\frac{\cos\theta}{\sin\theta}.$$

Now if $p=h+h'$, and $q=h-h'$; then $hh'=\dfrac{p^2-q^2}{4}$, $h^2+h'^2=\dfrac{p^2+q^2}{2}$;

$$\therefore\quad x=\tfrac{1}{4}.\left(\frac{p^2-q^2}{\sin\theta}-(p^2+q^2).\frac{\cos\theta}{\sin\theta}\right)$$

$$=\tfrac{1}{4}.\left\{p^2.\frac{1-\cos\theta}{\sin\theta}-q^2.\frac{1+\cos\theta}{\sin\theta}\right\}$$

$$=\tfrac{1}{4}.\left\{p^2.\tan\frac{\theta}{2}-q^2.\cot\frac{\theta}{2}\right\}.$$

In seconds, this correction

$$=\frac{1}{4.\sin 1''}.\left\{(p'\sin 1'')^2.\tan\frac{\theta}{2}-(q'\sin 1'')^2.\cot\frac{\theta}{2}\right\}$$

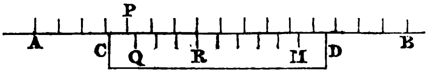
[If p', q', be the number of seconds in the angles p and q.]

$$=\frac{\sin 1''}{4}.\left\{p'^2.\tan\frac{\theta}{2}-q'^2.\cot\frac{\theta}{2}\right\}.$$

This reduction is not required when a Theodolite is used.

INSTRUMENTS USED IN SURVEYING.

82. **The Vernier** is a contrivance for subdividing equal divisions made on a straight line or a circle.



AB is a portion of a straight line, or a circle, which is divided by straight lines at right angles to it into any number of equal parts. CD, the Vernier, is another scale which slides along AB when AB is a straight line, and revolves round the centre of AB, when it is a circular arc.

If it be required to subdivide each of the divisions of AB into n equal parts, take $QM = n - 1$ of these parts, and divide QM into n equal parts by straight lines at right angles to it,—the magnitude of each of these parts will be $\frac{n-1}{n} \cdot a$, if a be the length of a division of AB.

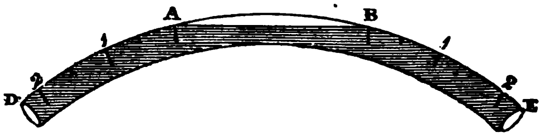
Let R, the r^{th} division from Q, coincide with a division on AB, then

$$PQ = PR - QR = ra - r \cdot \frac{n-1}{n} a = r \cdot \frac{a}{n};$$

and the length of PQ is known.

Ex. If the original divisions were an inch, and nine inches were divided into 10 parts on the Vernier, then if the extremity of a line to be measured came to Q, and the inches marked at P were p, if the third division of the Vernier coincided with a division on the scale, the length required would be $\left(p + \frac{3}{10}\right)$ inches.

83. THE SPIRIT-LEVEL is a glass tube *DABE* of circular bore, which is ground into the form of a circular arc of very large radius—sometimes 800 feet. It is then nearly filled with some fluid, and the ends are closed.



If the instrument be placed in a vertical plane, and the extremities *D* and *E* rest on a horizontal surface, the bubble (*AB*) of air left in the tube will be at the highest part of it, and if *E* be gradually raised the bubble will continually keep moving towards *E**.

If there be a plane of an instrument, such as a theodolite, which it is necessary to bring into a horizontal position, it is provided with two levels, as nearly equal to one another in every respect as possible, which are placed at right angles to each other, and permanently attached to the plane. The instrument-maker marks the positions of the bubbles when the plane is horizontal, and if therefore on any occasion the bubbles occupy these positions, the plane to which the levels are attached must be horizontal.

If the plane be inclined at any angle to the vertical and the positions of the bubbles are noted, then if at a second observation they occupy the same positions, the plane will have the same inclination to the vertical which it had before.

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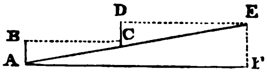
* The grinding the bore of a level is done with a plug of metal covered with emery. The elasticity of the glass, assisted probably by that of the metal of which the plug is formed, enables the workman, by means of pressure on the outside, to wear away any particular portion of the interior surface he chooses. If, after the grinding is finished, the bubble be found to be always of the same length, and to move through equal lengths of the tube for equal increments of inclination, the bore of the tube is uniform, and the form of the tube truly circular.

If l be the length through which the bubble moves in consequence of an increase of n'' in the inclination, (n being a small quantity,) the radius of the circular arc $= \frac{l}{n \,.\, \sin 1''} = \frac{l}{n} \times 20625$.

Ether is the best fluid with which levels can be filled; because in ether the bubble is found to come into a state of rest in the shortest time after a sudden displacement of it.

84. *To level between two points.*

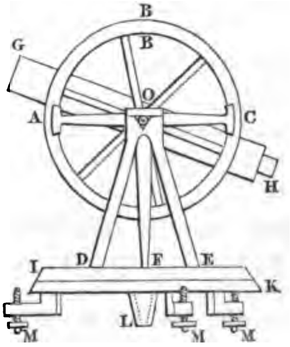
For the purpose of finding the altitude of one point above another point, a level is attached to a telescope, and so adjusted that when the bubble is at the middle of the level, the optical axis of the telescope is horizontal.



If there be two staffs set up, *AB*, *CD*, and when the instrument is at *A*, and its axis horizontal, the point *C* be seen on the cross wires, and if when the instrument is placed at *D* with its axis horizontal, *E* be seen on the cross wires, then by measuring *AB* and *CD*, we find $AB + CD$, or *EF*, the altitude of *E* above the horizontal line *AF* passing through *A*. The operation can be repeated as often as it may be necessary.

85. THE THEODOLITE. This is an instrument for measuring the angular altitudes of objects above a horizontal plane, and the horizontal angle which two objects subtend at the observer's eye.

The accompanying figure is an elevation,—that is, a projection on a vertical plane by lines perpendicular to the plane,—of a Theodolite of a very excellent construction, made by Robinson, of London.



The two circles *I* and *K* fit close to each other, the lower one having attached to it three horizontal bars, making angles of 120° with each other, on which the instrument rests. The feet of the instrument are three screws *MMM* working in the bars, and by moving these screws in one direction or the other, the planes of the circles *I* and *K*, which are parallel to each other, can be brought into a horizontal position.

The upper circle *I* is graduated, and revolves with the utmost possible nicety upon the lower circle by means of an axis attached to the upper circle, and working within the collar *L* which forms a part of the frame of the instrument. The circle *K* has two Verniers engraved upon it, and in the best instruments the divisions are read off by microscopes attached to the Verniers.

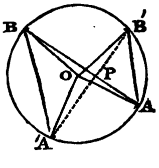
To the revolving circle, which is called the limb of the instrument, two levels at right angles to each other are attached.

To this circle also two stands are fixed, one of which *DOE* is here represented, supporting an axis parallel to the revolving circle *I*. To this axis a circle *ABC* is permanently attached at right angles to it, and a telescope is fastened to the circle, with its line of collimation perpendicular to the axis. With this axis the circle and attached telescope revolve in such a manner that the whole can turn completely round, without touching the limb of the instrument. The rim *BB* of the circle is graduated, and there are two Verniers, *A* and *C*, unconnected with the axis, but carried by a support which is seen to enter the circle *I* at *F*.

86. *To explain how the Theodolite is used.*

The two circles *I* and *K* are first rendered horizontal by lengthening or shortening the screw feet. The limb is then turned round upon the lower circle till the plane of the vertical circle passes through the object whose altitude is required, and the axis carrying the vertical circle is made to revolve, till the observer, on looking through the telescope, perceives the object on the cross wires with which the telescope is furnished. The graduations at the points of the rim of the vertical circle which are opposite to the beginning of the scales engraved on the Vernier plates are then read off.

Next, without touching the axis to which the telescope is attached, let the limb revolve through 180°. The plane of the vertical circle again passes through the object, but the direction of the telescope ($A'B'$) is now as much depressed below the horizon as it was before (AB) elevated above it. Let the vertical circle be turned with its axis in the direction $A'AB$ till the object is again seen on the cross wires of the telescope;—$A'B'$ is therefore brought into the same direction (AB) that it had at the first observation. The graduations are again read off at the Verniers, and the number of degrees marked on the arcs AA', BB' of the instrument, which are the arcs that have passed between the two Verniers, is four times the zenith distance of the object.



Let O be the real centre of the graduated circle, then twice the zenith distance is BPB', or APA'; join BA', AB';

$$\begin{aligned}
\therefore\ 4 \times \text{zenith distance} &= BPB' + APA' \\
&= PAB' + PB'A + PBA' + PA'B \\
&= BAB' + A'B'A + ABA' + B'A'B \\
&= 2BAB' + 2ABA' = BOB' + AOA';\quad \text{Euclid III. 20.}
\end{aligned}$$

= sum of differences of readings at the Verniers at the two observations.

It appears therefore that any errors arising from the axis of rotation of the circle not coinciding exactly with the centre of its graduation are in this manner avoided. (Astronomy, Art. 17.)

To observe the horizontal angle between two objects, the limb of the Theodolite is made horizontal, and the angle is noted, (by four readings off, as in the last case), through which the limb revolves to bring the two objects successively on the cross wires of the telescope.

87. **The Repeating Circle** was the instrument used in the French surveys for observing the angles. The observations, however, which are made by it, are of questionable value, for this reason,—that although the errors of imperfect graduation may possibly, as they are probably, destroyed by repeating the observations and taking the mean of them; yet, for any thing that can be known to the contrary, the errors arising from the inability of the observer to distinguish the position of a point with perfect exactness, may have been going on all the while continually accumulating.

The construction also of the Repeating Circle is such, that it is scarcely possible that instrumental errors arising from instability can be avoided entirely, or nearly so.

CHAPTER VII.

ON (I.) THE SMALL CORRESPONDING VARIATIONS OF THE PARTS OF A SPHERICAL TRIANGLE, AND (II.) THE CONNEXION EXISTING BETWEEN SOME FORMULÆ IN SPHERICAL TRIGONOMETRY, AND ANALOGOUS FORMULÆ IN PLANE TRIGONOMETRY.

I. **The** process of Differentiation can be applied to determine the errors introduced in determining the other parts of a Spherical Triangle from three given parts, when one of the given parts is affected by a small known error.

88. *If* C *and* c *remain constant, the corresponding variations* δa, δb *of the other sides are connected by the equation,*

$$\delta \mathrm{b} \,.\, \cos \mathrm{A} + \delta \mathrm{a} \,.\, \cos \mathrm{B} = 0.$$

Considering C and c constant, we have by differentiating the formula,

$$\operatorname{Cos} C \,.\, \sin a \,.\, \sin b = \cos c - \cos a \,.\, \cos b,$$

$$\cos C \,.\, \{\cos a \,.\, \sin b + d_a b \,.\, \cos b \,.\, \sin a\} = \sin a \,.\, \cos b + d_a b \,.\, \sin b \,.\, \cos a\,;$$

$$\therefore\ \cos C \,.\, \{\delta a \,.\, \cos a \,.\, \sin b + \delta b \,.\, \cos b \,.\, \sin a\}$$

$$= \delta a \,.\, \sin a \,.\, \cos b + \delta b \,.\, \sin b \,.\, \cos a, \quad \text{nearly};$$

$$\therefore\ 0 = \delta a \,.\, \{\sin a \,.\, \cos b - \cos C \,.\, \sin b \,.\, \cos a\}$$

$$+ \delta b \,.\, \{\sin b \,.\, \cos a - \cos C \,.\, \sin a \,.\, \cos b\}$$

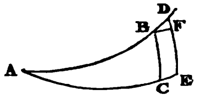
$$= \delta a \,.\, \cos B \,.\, \sin c + \delta b \,.\, \cos A \,.\, \sin c\,; \quad \text{by (iv)}.$$

$$\therefore\ 0 = \delta a \,.\, \cos B + \delta b \,.\, \cos A.$$

This method is always applicable. A small spherical triangle is however frequently treated as a plane triangle after the following manner, particularly in establishing formulæ for calculating the corrections used in Astronomical investigations.

89. ***If* c, *the side opposite the right angle in a right-angled spherical triangle, receive a small known increment, to determine the corresponding increments of the sides enclosing the right angle.***

Let $AB = c$, $BD = \delta c$, $\angle BAC = \omega$. Draw DE perpendicular to AE, and let BF be an arc of a small circle which has the same pole as ACE;—BF is therefore parallel to AE, and perpendicular to DE.



Now, considering BDF as a small plane triangle,

$$\frac{BF}{\delta c} = \cos DBF = \sin ABC.$$

(For since $\angle CBF = 90^0$, $\angle ABC = 90^0 - \angle FBD$);

$$\therefore\ BF = \delta c\,.\,\frac{\cos\omega}{\cos BC}, \quad \text{by Napier's rules.}$$

$$\therefore\ \delta(AC), \text{ or } CE, = \frac{BF}{\cos BC}, \text{ Art. 15.} = \delta c\,.\,\frac{\cos\omega}{(\cos BC)^2} \ldots (1).$$

Again,

$$\delta(BC) = FD = \delta c\,.\sin DBF$$

$$= \delta c\,.\cos ABC = \delta c\,.\sin\omega\,.\cos AC\,.\ldots\ldots.(2).$$

Cor. If ω be constant, it will be seen that

$$\left\{\begin{array}{l}\delta(AC) \text{ has its greatest value, when } BC \text{ is the greatest.}\\ \delta(AC) \ \ldots\ldots\ldots \ \text{least} \ \ldots\ldots, \text{ when } BC = 0^0.\end{array}\right.$$

$$\left\{\begin{array}{l}\delta(BC) \text{ has its greatest value, when } AC = 0^0, \text{ or } 180^0.\\ \delta(BC) \ \ldots\ldots\ldots \ \text{least} \ \ldots\ldots, \text{ when } AC = 90^0.\end{array}\right.$$

This result shews, that if the Sun's daily motion (δc) in longitude be nearly constant, as it is in fact, his greatest daily change in Right Ascension, $\delta(AC)$, is at the solstices, and his greatest daily change in Declination, $\delta(BC)$, is at the equinoxes; and his least daily changes in Right Ascension, and in Declination, are at the equinoxes and solstices respectively.

Also; since when BC and AC are small angles, the variation in magnitude of their cosines is then the least, it appears from (1) and (2) that the daily change both in right ascension and in declination is more nearly proportional to the change in longitude, (that is, to the time), when the Sun is near the equinoxes than when he is in any other part of his apparent orbit. (See Hymers' Astronomy, Art. 141.)

II. 90. *From the formula,* $\frac{\sin \mathrm{A}}{\sin \mathrm{B}} = \frac{\sin \mathrm{a}}{\sin \mathrm{b}}$, *which is true for a* spherical *triangle, to deduce an analogous formula for the case of a* plane *triangle.*

Let a', b' be the lengths of the arcs subtending the angles A and B; r the radius of the sphere.

$$\text{Then } \frac{\sin A}{\sin B} = \frac{\sin \frac{a'}{r}}{\sin \frac{b'}{r}} = \frac{\frac{a'}{r} - \frac{1}{2.3}\cdot\frac{a'^3}{r^3} + \dots}{\frac{b'}{r} - \frac{1}{2.3}\cdot\frac{b'^3}{r^3} + \dots}$$

$$= \frac{a'}{b'}\cdot\frac{1 - \frac{1}{2.3}\cdot\frac{a'^2}{r^2} + \dots}{1 - \frac{1}{2.3}\cdot\frac{b'^2}{r^2} + \dots}$$

And if a' and b' are indefinitely small compared with the radius of the sphere, this formula becomes

$$\frac{\sin A}{\sin B} = \frac{a'}{b'};$$

a property of plane triangles.

91. *In any spherical triangle,*

$$\operatorname{Tan}\tfrac{1}{2}(A - B) = \frac{\sin\frac{1}{2}(a - b)}{\sin\frac{1}{2}(a + b)}\cdot\cot\frac{C}{2};$$

to deduce the analogous formula in Plane Trigonometry.

As in the last example,

$$\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)} = \frac{\frac{1}{2}(a'-b')}{\frac{1}{2}(a'+b')},$$

(when the sides are indefinitely small compared with r),

$$= \frac{a'-b'}{a'+b'};$$

$$\therefore\ \text{Tan}\,\tfrac{1}{2}(A-B) = \frac{a'-b'}{a'+b'}\,.\cot\frac{C}{2}.$$

Now in a plane triangle, $\dfrac{\sin A}{\sin B} = \dfrac{a'}{b'}$;

$$\therefore\ \frac{a'-b'}{a'+b'} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$$

and $\tan\frac{1}{2}(A+B) = \tan\frac{1}{2}.(180^0 - C) = \cot\dfrac{C}{2}$;

$$\therefore\ \text{Tan}\,\tfrac{1}{2}(A-B) = \frac{a'-b'}{a'+b'}\,.\cot\frac{C}{2},$$

which agrees with the result just obtained.

Cor. In the same manner

$$\text{Tan}\,\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\,.\cot\frac{C}{2},$$

will be found to reduce to $\text{Tan}\,\frac{1}{2}(A+B) = \cot\dfrac{C}{2}$, when the sides of the spherical triangle are indefinitely small compared with the radius of the sphere.

92. After the same manner, the formula

$$\text{Cos}\,A = \frac{b'^2 + c'^2 - a'^2}{2b'c'},$$

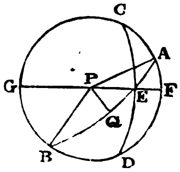
may be deduced from

$$\text{Cos}\,A = \frac{\cos a - \cos b\,.\cos c}{\sin b\,.\sin c}.$$

93. Prob. *If two arcs of great circles intersect each other in a small circle, the product of the tangents of the semi-segments of the one is equal to that of the tangents of the semi-segments of the other.*

Let AB, CD be the arcs intersecting in E, a point within the small circle whose pole is P; $FEPG$ an arc of a great circle through E and P; PQ perpendicular to AB.

Then AB is bisected in Q, and APQ, BPQ are equal triangles in every respect.



$$\text{Also} \quad PA = PB = PF = PG.$$

By Napier's rules,

$$\operatorname{Cos} PE = \cos EQ \,.\, \cos PQ, \quad \text{and} \quad \operatorname{Cos} PA = \cos AQ \,.\, \cos PQ;$$

$$\therefore \frac{\cos EQ - \cos AQ}{\cos EQ + \cos AQ} = \frac{\cos PE - \cos PA}{\cos PE + \cos PA};$$

$$\therefore \tan \tfrac{1}{2}(AQ+EQ) \,.\, \tan \tfrac{1}{2}(AQ-EQ) = \tan \tfrac{1}{2}(PA+PE) \,.\, \tan \tfrac{1}{2}(PA-PE),$$

$$\text{or} \quad \tan \tfrac{1}{2} BE \,.\, \tan \tfrac{1}{2} EA = \tan \tfrac{1}{2} GE \,.\, \tan \tfrac{1}{2} EF = \tan \tfrac{1}{2} CE \,.\, \tan \tfrac{1}{2} ED,$$

in like manner.

Cor. Expanding these tangents by the formula

$$\operatorname{Tan} \theta = \frac{\sin \theta}{\cos \theta} = \frac{\theta - \dfrac{\theta^3}{2\,.\,3} + \ldots}{1 - \dfrac{\theta^2}{2} + \ldots}$$

$$= \left\{\theta - \frac{\theta^3}{2\,.\,3} + \ldots\right\} . \left\{1 + \frac{\theta^2}{2} + \ldots\right\}.$$

we shall have, if a, b represent the lengths of the segments of the arc AB, and c, d represent the lengths of the segments of the arc CD, and these quantities be indefinitely small compared with the radius of the sphere,

$$ab = cd;$$

which agrees with Euclid III. 35.

CHAPTER VIII.

ON REGULAR SOLIDS.

94. Def. (1). A *polyhedron* is a solid bounded by plane rectilinear figures.

If the bounding surfaces be any similar and equal regular rectilinear figures, the polyhedron is called a *regular* polyhedron.

(2). A *tetrahedron* is bounded by four equal and equilateral triangles.

(3). A *hexahedron*, or *cube*, is bounded by six equal squares.

(4). An *octahedron* is bounded by eight equal and equilateral triangles.

(5). A *dodecahedron* is bounded by twelve equal and equilateral pentagons.

(6). An *icosahedron* is bounded by twenty equal and equilateral triangles.

It will be proved hereafter that no more regular polyhedrons exist than these five.

95. *In any regular polyhedron, if*

F = *number of faces,*

E = *number of edges,*

S = *number of solid angles,*

m = *number of sides in each face;*

then $2\mathrm{E} = \mathrm{mF}$, *and* $\mathrm{S} + \mathrm{F} = \mathrm{E} + 2$.

(1). Since every edge is made by two sides, the whole number of sides in the polyhedron is $2E$, and this

$$= \text{number of faces} \times \text{number of sides in a face},$$

$\therefore\ 2E = mF.$

(2). Take any point within the polyhedron as the centre of a sphere whose radius is r, and join it with each of the angular points of the polyhedron. Let the points in which these lines cut the surface of the sphere be joined by arcs of great circles, so that the surface of the sphere is divided into as many polygons as the polyhedron has faces. The Area of one of these polygons described on the sphere,

$$= \frac{\pi r^2}{180^0} \cdot \left\{ \begin{array}{l} \text{angles of the polygon} \\ - (\text{number of sides of polygon} - 2) \,.\, 180^0 \end{array} \right\}.$$

Art. 64. Cor. 2.

$\therefore$ Area of all these polygons

$$= \frac{\pi r^2}{180^0} \cdot \left\{ \begin{array}{l} \text{angles of all the polygons on the sphere} \\ - (\text{number of all the sides} - 2F) \,.\, 180^0 \end{array} \right\}$$

$$= \frac{\pi r^2}{180} \cdot \{S \,.\, 360 - (E - F) \,.\, 360\}$$

$$= 2\pi r^2 \,.\, (S - E + F).$$

But Area of all the polygons = area of sphere $= 4\pi r^2$;

$$\therefore\ S - E + F = 2,$$

$$\text{And}\ S + F = E + 2;$$

(A result evidently true whether the polyhedron be regular or irregular.)

96. *The sum of all the plane angles which form the solid angles of a polyhedron* $= (S - 2) \,.\, 360^0$.

For the plane angles = sum of all the interior angles of each face.

$$= F \,.\, (m - 2) \,.\, 180^0 \quad \text{Eucl. I. 32. Cor. 1.}$$

$$\left. \begin{array}{l} = (E - F) \,.\, 360^0 \\ = (S - 2) \,.\, 360^0. \end{array} \right\} \ \text{Art. 89.}$$

97. *To prove that*

$$S = \frac{4m}{2(m+n)-mn}, \quad F = \frac{4n}{2(m+n)-mn}, \quad E = \frac{2mn}{2(m+n)-mn},$$

where m *and* n *are respectively the number of sides in every face, and the number of plane angles in every solid angle.*

Since every face has m plane angles,

$\therefore$ number of angles forming all the solid angles $= mF$, and $\therefore = Sn$.

Hence $\quad Sn = mF = 2E, \quad$ Art. 89.

and $\quad S + F = E + 2, \quad$ Art. 89.

$$\therefore\ 2 = S + F - E = S\,.\left(1 + \frac{n}{m} - \frac{n}{2}\right);$$

$$\therefore\ S = \frac{4m}{2(m+n)-mn},$$

$$\therefore\ F = \frac{4n}{2(m+n)-mn},$$

$$E = \frac{2mn}{2(m+n)-mn}.$$

98. *There can be but five regular polyhedrons.*

In any regular polyhedron, m, n, S, F, E must each be positive integers.

In order that the values of S, F, E, obtained in the last Article, may be positive, $2(m+n)$ must be greater than mn,—and that they may each be integral, $4m$, $4n$, $2mn$ must be severally divisible by $2(m+n)-mn$.

Now if $2(m+n)$ be greater than mn,

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}, \quad \text{or} \quad \frac{1}{m} > \frac{1}{2} - \frac{1}{n}.$$

But n cannot be less than 3;

$$\therefore\ \frac{1}{m} \text{ cannot be so small as } \frac{1}{2} - \frac{1}{3}, \text{ or } \frac{1}{6};$$

Therefore, since m cannot be less than 3, it can only be 3, 4, or 5.

Similarly, since $\frac{1}{n} > \frac{1}{2} - \frac{1}{m}$, and m cannot be less than 3;

$\therefore$ the values of n can only be 3, 4, or 5.

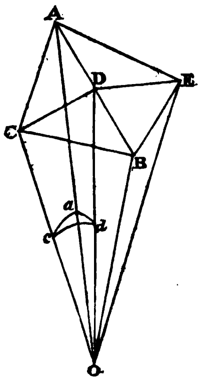
On trial it will appear that the only values of m and n which satisfy all the required conditions are the following. Each regular solid takes its name from the number of its plane faces.

m.	*n.*	*S.*	*E.*	*F.*	Name of the Regular Solid.
3	3	4	6	4	Tetrahedron. (Regular pyramid.)
4	3	8	12	6	Hexahedron. (Cube.)
3	4	6	12	8	Octahedron.
5	3	20	30	12	Dodecahedron.
3	5	12	30	20	Icosahedron.

Cor. If $2(m+n) = mn$, S, E, and F, become infinite quantities, and the solid itself becomes a sphere.

99. *If* I *be the inclination of two contiguous faces,*

then, $$\operatorname{Sin}\frac{I}{2} = \frac{\cos\frac{180^\circ}{n}}{\sin\frac{180^\circ}{m}}, \quad \text{or} = \frac{\cos\left(\frac{S}{E}\cdot 90^\circ\right)}{\sin\left(\frac{F}{E}\cdot 90^\circ\right)}.$$



Let C and B be the centres of two adjacent faces whose common edge is AB; bisect AB in D, and join CA, CB. CD and ED are manifestly perpendicular to AB, and $\therefore \angle CDE = I$.

In the plane CDE, draw CO and EO at right angles to CD and ED respectively; let these lines meet in O; join OA, OB, OD.

About O as centre describe a spherical surface which is cut by the planes AOD, DOC, COA in ad, dc, ca.

Now since *AB* is perpendicular to *CD* and to *ED*, it is perpendicular to the plane *CDE*, and therefore the plane *AOB*, in which *AB* lies, is perpendicular to the plane *COE*;

$$\therefore\ \angle adc \text{ is a right angle,}$$

And

$$\angle cad = 180^0 \div \text{number of edges which meet in a solid angle}$$

$$= 180^0 \div \text{number of plane angles which meet in a solid angle,}$$

$$= \frac{180^0}{n};$$

$$\angle acd = \left\{\begin{matrix}\tfrac{1}{2}\angle \text{ which each side of a plane face} \\ \text{subtends at the centre of the face}\end{matrix}\right\} = \frac{1}{2}\cdot\frac{360^0}{m} = \frac{180^0}{m}.$$

Now by Napier's Rules,

$$\text{Cos}\,\angle cad = \cos dc\,.\,\sin\angle acd,$$

$$\therefore\ \cos\frac{180^0}{n} = \cos dc\,.\,\sin\frac{180^0}{m};$$

$$\text{and}\quad \cos dc = \cos DOC = \cos\frac{COE}{2}$$

$$= \cos\tfrac{1}{2}\,(180^0 - CDE) = \sin\frac{I}{2};$$

$$\therefore\ \text{Sin}\,\frac{I}{2} = \frac{\cos\dfrac{180^0}{n}}{\sin\dfrac{180^0}{m}}.$$

Again, Since $n = \dfrac{2E}{S}$, and $m = \dfrac{2E}{F}$, Art. 89,

$$\therefore\ \text{Sin}\,\frac{I}{2} = \frac{\cos\left(\dfrac{S}{E}\cdot 90^0\right)}{\sin\left(\dfrac{F}{E}\cdot 90^0\right)}.$$

100. *To find the radius of a sphere inscribed in a regular polyhedron.*

$OC = OE$, $(=r)$, is this radius. Let $AB = 2a$.

$$\text{Then } CD = AD \cdot \cot ACD = a \cdot \cot ACD = a \cdot \cot \frac{180^\circ}{m};$$

$$\text{And } r = CD \cdot \tan CDO$$

$$= CD \cdot \tan \frac{I}{2}$$

$$= a \cdot \tan \frac{I}{2} \cdot \cot \frac{180^\circ}{m}.$$

101. *To find the radius of a sphere described about a regular polyhedron.*

$OA = OB$, $(=R)$, is this radius.

$$\text{And } r = R \cdot \cos ac = R \cdot \cot acd \cdot \cot cad = R \cdot \cot \frac{180^\circ}{m} \cdot \cot \frac{180^\circ}{n}.$$

$$\therefore\ R = r \cdot \tan \frac{180^\circ}{m} \cdot \tan \frac{180^\circ}{n}$$

$$= a \cdot \tan \frac{I}{2} \cdot \tan \frac{180^\circ}{n}. \quad \text{Art. 94.}$$

102. If $2a$, or an edge of the polyhedron, be given, and we apply the formulæ of the two preceding Articles to determine the values R and r in the five regular polyhedrons, we shall arrive at the same values for both the hexahedron and the octahedron, and for both the dodecahedron and the icosahedron. Hence each of these pairs of regular polyhedrons may be inscribed in, or circumscribed about, the same sphere;—each of the polyhedrons in the pair having the same edge.

103. *To find the volume of a regular polyhedron.*

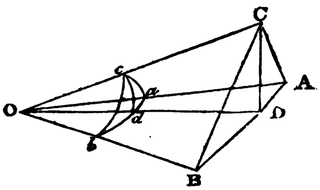
From O in fig. Art. 93, draw OA, OB, &c. to all the angles of the solid, which will thus be divided into F pyramids, whose common altitude is r, and common base, being the area of a face, $= m . a . CD = m . a . r . \cot \frac{I}{2}$; Art. 94.

$$\therefore \text{ whole volume} = \frac{1}{3} . F . m . a . r^2 . \cot \frac{I}{2} .$$

$$= \frac{1}{3} . F . m . \tan \frac{I}{2} . \left(\cot \frac{180^\circ}{m}\right)^2 . a^3 . \quad \text{Art. 94.}$$

Cor. Therefore, Volume $\propto a^3$, in similar polyhedrons.

104. *In a parallelopiped, given the three edges which meet, and the angles between them, to find the altitude, surface, and volume, of the solid.*



Let AO, BO, CO be the three edges meeting in the point O, and let

$\angle BOC = A$, $\angle AOC = B$, $\angle AOB = C$.

$OA = a$, $\quad OB = b$, $\quad OC = c$.

$S = \frac{1}{2}(A + B + C)$

CD perpendicular to plane AOB. Join OD, BD, DA.

About O describe a spherical surface which is cut by the planes AOB, BOC, COA, DOC, in ab, bc, ca, dc.

Then, the angles at d being right angles,

$$\sin dc = \sin ac . \sin dac$$

$$\text{or} \quad \sin DOC = \sin B . \sin dac;$$

$$\text{And } \operatorname{Sin} dac = \frac{2}{\sin B . \sin C} . \sqrt{\{\sin S . \sin(S - A) . \sin(S - B) . \sin(S - C)\}},$$

by (xix.);

$$\therefore DC = c . \sin DOC = \frac{2c}{\sin C} . \sqrt{\{\sin S . \sin(S - A) . \sin(S - B) . \sin(S - C)\}}.$$

The Surface, $= 2 . \{bc . \sin A + ac . \sin B + ab . \sin C\}$.

$$\text{The Volume} = \left\{ \begin{array}{l} \text{Volume of rectangular parallelopiped on} \\ \quad \text{the same base, and of the same altitude,} \end{array} \right.$$

$= \text{area of base} \times \text{altitude}$.

Euclid XI. 31.

$$= 2abc . \sqrt{\{\sin S . \sin (S-A) . \sin (S-B) . \sin (S-C)\}}.$$

105. *The same things being given, to determine the diagonal which passes through the solid angle* O.

Let D be the diagonal required, and let OdD in the fig. Art. 98, now be the direction of the diagonal of the face AOB. Let this diagonal $= d$.

Then $d^2 = a^2 + b^2 + 2ab . \cos C$,

$$\begin{aligned} \text{and}\quad D^2 &= c^2 + d^2 - 2cd . \cos (180^0 - DOC), \\ &= c^2 + d^2 + 2cd . \cos DOC, \\ &= a^2 + b^2 + c^2 + 2cd . \cos DOC + 2ab . \cos C. \end{aligned}$$

Now

$$\operatorname{Cos} DOC = \cos cd = \cos ac . \cos ad + \sin ac . \sin ad . \cos bac$$

$$= \cos B . \cos ad + \sin B . \sin ad . \frac{\cos A - \cos B . \cos C}{\sin B . \sin C}.$$

$$= \frac{1}{\sin C} . \{\cos B . \sin C . \cos ad - \cos B . \cos C . \sin ad + \cos A . \sin ad\}$$

$$= \frac{1}{\sin C} . \{\cos B . \sin (C - ad) + \cos A . \sin ad\}$$

$$= \frac{1}{\sin C} . \{\cos B . \sin BOD + \cos A . \sin AOD\}.$$

But $\frac{a}{d} = \frac{\sin BOD}{\sin C}$, and $\frac{b}{d} = \frac{\sin AOD}{\sin C}$;

$$\therefore \ \text{Cos}\, DOC = \frac{a \,.\, \cos B + b \,.\, \cos A}{d}.$$

And $D^2 = a^2 + b^2 + c^2 + 2bc \,.\, \cos A + 2ac \,.\, \cos B + 2ab \,.\, \cos C$.

Cor. 1. At the solid angle C,

The cosine of the angle between c and $b = - \cos COB = - \cos A$,

.... c and a $= - \cos B$

.... a and b $= \cos C$;

Therefore the diagonal through C

$$= a^2 + b^2 + c^2 - 2bc \,.\, \cos A - 2ac \,.\, \cos B + 2ab \,.\, \cos C.$$

Similarly, the diagonals through the other solid angles B, and A, are,

$$a^2 + b^2 + c^2 - 2bc \,.\, \cos A + 2ac \,.\, \cos B - 2ab \,.\, \cos C,$$

$$a^2 + b^2 + c^2 + 2bc \,.\, \cos A - 2ac \,.\, \cos B + 2ab \,.\, \cos C.$$

Cor. 2. Hence it appears that the sum of the squares of the four diagonals

$$= 4\,(a^2 + b^2 + c^2).$$

EXAMPLES.

1. If two arcs of great circles intersect at right angles in a point in the circumference of a small circle, then if one of them touch the small circle, the other bisects it.

2. On the surface of a sphere draw a great circle passing through a given point, and touching a given small circle.

IN A RIGHT-ANGLED SPHERICAL TRIANGLE, C BEING THE RIGHT ANGLE.

3. $2 \cos c = \cos (a + b) + \cos (a - b)$.

4. $\text{Tan} \frac{1}{2} (c + a) \,.\, \tan \frac{1}{2} (c - a) = (\tan \frac{1}{2} b)^2$.

5. $(\text{Sin} \frac{1}{2} c)^2 = (\sin \frac{1}{2} a \,.\, \cos \frac{1}{2} b)^2 + (\cos \frac{1}{2} a \,.\, \sin \frac{1}{2} b)^2$.

6. $(\text{Sin}\, b \,.\, \cos a)^2 = \sin (c + a) \,.\, \sin (c - a)$.

7. If there be two right-angled spherical triangles ABC, $AB'C'$, having the angle A common to both,

$$\text{Tan} \tfrac{1}{2} (a + a') \,.\, \tan \tfrac{1}{2} (c - c') = \tan \tfrac{1}{2} (a - a') \,.\, \tan \tfrac{1}{2} (c + c').$$

8. If α, β be the arcs drawn from C, respectively perpendicular to c, and bisecting c;

$$\text{Cot}\, \alpha = \sqrt{\{(\cot a)^2 + (\cot b)^2\}}\,; \quad \text{Cot}\, \beta = \frac{\cos a + \cos b}{\sqrt{\{(\sin a)^2 + (\sin b)^2\}}},$$

$$\text{Sin} \frac{c}{2} = \frac{\sin \beta}{\sqrt{\{1 + (\sin \alpha)^2\}}}.$$

IN ANY SPHERICAL TRIANGLE.

9. If each of the three sides be quadrants, and α, β, γ be the distances of a point within the triangle from the angular points;

$$(\cos \alpha)^2 + (\cos \beta)^2 + (\cos \gamma)^2 = 1.$$

10. If d be the length of the arc which bisects C, and is terminated by the opposite side,

$$\operatorname{Tan} d \,.\, \sin(a+b) = 2 \sin a \,.\, \sin b \,.\, \cos \tfrac{1}{2} C.$$

11. If a and b be nearly equal,

$$a = \tfrac{1}{2}(a+b) + \tan \tfrac{1}{2}(a+b) \,.\, \tan \tfrac{1}{2}(A-B) \,.\, \cot \tfrac{1}{2}(A+B), \text{ very nearly.}$$

12. If one angle of a triangle, plane or spherical, be equal to the sum of the other two angles, the greatest side is double of the distance of its middle point from the opposite angle.

13. Find the locus of the vertices of all right-angled spherical triangles which have the same hypothenuse; and from the equation prove that, when the radius of the sphere is infinite, the locus is a circle.

AREAS.

14. By an arc drawn from an angle to the side opposite, divide a given triangle into two others whose areas are in a given ratio.

15. The sides of a spherical triangle are each $111^0, 28'$; find its angles, and shew that its area $= \frac{1}{4}$ surface of the sphere. ($111^0 . 28' = 2 \times 55^0, 44'$, and $\operatorname{Tan} \frac{1}{2}(55^0, 44') = \sqrt{2}$).

16. If E be the spherical excess,

$$\operatorname{Cot} \tfrac{1}{2} E = \frac{\cot \frac{1}{2} a \,.\, \cot \frac{1}{2} b + \cos C}{\sin C},$$

$$\text{or } = \frac{1 + \cos a + \cos b + \cos c}{2\sqrt{\{\sin S \,.\, \sin(S-a) \,.\, \sin(S-b) \,.\, \sin(S-c)\}}}$$

$$\operatorname{Cos} \tfrac{1}{2} E = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{1}{2} a \,.\, \cos \frac{1}{2} b \,.\, \cos \frac{1}{2} c}.$$

In seconds,

$$\tfrac{1}{2} E = \frac{\tan \frac{1}{2} b \,.\, \tan \frac{1}{2} c \,.\, \sin A}{\sin 1''} - \frac{(\tan \frac{1}{2} b \,.\, \tan \frac{1}{2} c)^2 \,.\, \sin 2A}{\sin 2''}$$

$$+ \frac{(\tan \frac{1}{2} b \,.\, \tan \frac{1}{2} c)^3 \,.\, \sin 3A}{\sin 3''} - \ldots$$

17. Determine the area of a spherical triangle from the data a, b, C; and shew that if a and b be constant, and also $a+b$ be less than π, the area admits of a maximum value.

18. The angles of a spherical triangle, of which the area is $\frac{7}{12}.\pi r^2$, where r is the radius of the sphere, form an arithmetic progression whose common difference is 45^0. Find them.

19. If the sides AC, BC of a triangle be produced to D and E, points such that $\tan\frac{1}{2}AC.\tan\frac{1}{2}BC=\tan\frac{1}{2}DC.\tan\frac{1}{2}EC$, and DE be joined by an arc of a great circle, the triangles ABC, CDE are of equal area.

20. If S be the surface of a spherical triangle whose angles are each 120^0, and S' that of its polar triangle,

$$\text{Tan}\,\tfrac{1}{3}S : \text{Tan}\,\tfrac{1}{3}S' = 6\sqrt{2}+\sqrt{3} : 2\sqrt{2}-\sqrt{3}.$$

21. If a be one of the n sides of a regular spherical polygon, its surface (S) may be found from the equation

$$\text{Cos}\,\frac{\pi-\frac{1}{2}S}{n}=\cos\frac{\pi}{n}.\sec\frac{a}{2}.$$

22. The spherical excess of a triangle is $1''\cdot5$. Find its area, the radius of the Earth being taken to be 7757 miles.

23. If the three sides of a spherical triangle measured on the Earth's surface, be 12, 16, and 18 miles, find the spherical excess.

24. A plane triangle whose sides are a, b, c, is placed in a sphere of radius r. Prove that the angle between the arcs of the great circles of which a and b are the chords is a right angle, if $2r.\sqrt{(a^2+b^2-c^2)}=ab$.

25. If P be the pole of the small circle circumscribing a triangle ABC, prove that $\angle APB$ is double of the angle between the chords of AC and BC.

26. If i be the inclination of a plane to the horizon, and a the inclination of a line in it to the intersection of the plane with the horizontal plane, the inclination θ of the line to the horizon will be found from the equation

$$\operatorname{Sin}\theta = \sin i \,.\, \sin a.$$

27. If Z be the zenith, K the pole of the limb (which is *not exactly* horizontal) of a theodolite, and S be an object whose azimuth is observed, the error is known from the equation,

$$\angle SKZ - \angle SZK = KZ \,.\, \cot SK \,.\, \sin SZK.$$

28. If Z be the zenith, K the pole of the circle of a theodolite, which is *not exactly* vertical, and KZ be produced to CZ till KQ is a quadrant, then if S be an object whose zenith distance is to be observed, the error of observation is known from the equation,

$$SQ - SZ, = \cot SQ \,.\, \sin \tfrac{1}{2}\, QZ.$$

29. Given two sides, a and b, of a triangle, plane or spherical, and the included angle C, to find the variation produced in A corresponding to a small given variation in C.

30. If a solid be bounded by plane figures, of which some have an odd, and some have an even, number of sides, shew that there must be an even number of those faces which have an odd number of sides.

31. If r, R be the radii of the spheres described within and about a regular tetrahedron, r', R' the radii of the spheres to which the edges, and one face and the planes of the three others produced, are respectively tangents, prove that

$$r' = \sqrt{rR}, \quad \text{and } R' = \sqrt{2rR}.$$

32. Given the six edges of a triangular pyramid, to find its volume.

33. Of all triangular pyramids of given volumes, the regular tetrahedron has the least surface.